ITERATIVE DESIGN AND COMPARISON OF LEARNING SYSTEMS FOR REFLECTION IN TWO DIMENSIONS

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Abstract

This research involved the iterative design and comparison of learning systems aiming to mediate between students' personal knowledge of reflection and conventional mathematical knowledge.

The study comprised a two-phase teaching experiment. In the first phase, four learning systems were developed using an iterative methodology that cycled between design of computer-based tools and observation of students interacting with them. Each learning system was structured according to a *filling-outwards* or a *filling-inward* instructional approach and incorporated a computer microworld based on either dynamic Euclidean geometry (*DEG*) or multiple-turtle geometry (*MTG*). Learning systems were intended (i) to help students build from views of reflection based on internal (intra) properties of two-dimensional figures to views taking account of external (inter) relationships between figures and (ii) to emphasise a functional perspective on the transformation.

In the second phase, analysis focused on how meanings for reflection evolved as six students' interacted in each system. Results suggested that, in all four systems, students developed meanings by coordinating intra and interfigural analyses while they built computational models of reflection. Microworld tools had a role in mediating all aspects of students' activities: with DEG tools, reflection tended to be represented as a correspondence relationship based on perpendicular distances; MTG tools led to expression of reflection as a mapping of one set of turtles onto another and emphasised equal turns and distances. In all learning systems, mathematical meaning-making involved forging connections between general models of reflection and physical movements of screen objects. Additionally, students using MTG tools gave more meaning to their models by connecting feedback to imaginary social practices. The impact of instructional approach on learning trajectories was also mediated by microworld tools. Specific effects associated with filling-inwards and filling-outwards interventions were identified, but were limited to particular tasks and the tools used to negotiate them.

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Chapter 1

Introduction

> **reflection** ...(in a line) A transformation, involving a *mirror line* or *axis l*, such that a line segment joining a point to its image is perpendicular to l and has its midpoint on l. Reflection of points of the plane in the line with equation $y = x\tan\theta$ maps the points with co-ordinates (x, y) onto the point
>
> $$(x\cos 2\theta + y\sin 2\theta, x\sin 2\beta - y\cos 2\theta).$$
>
> In particular, reflection in the y-axis maps (x, y) onto (–x, y). ...*See also* symmetry.
>
> The Penguin Dictionary of Mathematics (1998; p.361)

The dictionary definition of reflection illustrates some of the complexities involved in mathematics learning. First, it describes how reflection is a transformation, suggesting that understanding reflection requires understanding of the group of objects to which it belongs. Another view can be found in the guidelines of the English mathematics curriculum (DEE, 1999): transformations as objects are not mentioned until Key Stage 3 (which begins after students have reached the age of 11 years), whereas aspects of reflection are to be covered in all four Key Stages, beginning, in Key Stage 1, with the recognition of reflective symmetry in familiar 2-D shapes and patterns. The curriculum guidelines hence indicate that students should learn about reflection as a property before, and separately from, reflection as a transformation.

The dictionary definition goes on to define the transformation reflection with reference to two different formal systems of mathematics, with terms associated with Euclidean geometry (perpendicular, midpoints) as well as trigonometric functions employed to describe reflection in a co-ordinate geometry context. Here, what is worth noting is how concise are both the definitions. They do not list all the properties that could be associated with the reflection transformation. Instead they express a general relationship from which all other properties might be derived.

In school mathematics, reflection is also represented in both Euclidean and co-ordinate geometry contexts. The message of the curriculum is of a gradual build up towards the kind of generalisability expressed in the dictionary definition, so students are expected to learn that reflections are specified by a mirror line "... at first using a line parallel to an axis, then a mirror line such as $y = x$ or $y = -x$" (DEE, 1999; p.50).

Mathematical dictionaries and school curriculum documents have in common that they contain specifications related to institutionalised knowledge of mathematics. Dictionary definitions of reflection can be viewed as resources to aid the construction of mathematical meanings. They tend to present general statements from which more specific cases can be derived. Curriculum documents prescribe the mathematical knowledge that school-students are supposed to develop and, to a substantial degree, determine the ways in which it will be encountered in the school subject. In contrast to the dictionary definition, the English mathematics curriculum suggests particular cases are presented before learners confront more general models.

While both kinds of institutionally accepted descriptions of knowledge are written to be precise and complete, understanding the meanings that individuals attach to the concepts defined involves understanding the whole range of images, representations, properties and situations that they evoke, as well as the inter-relationships between them. An idea like reflection is not confined to the mathematics classroom. It is encountered in a variety of other situations and has a range of physical, visual, cognitive and meta-cognitive associations. For the 12-13 year-old students involved in this study, reflection is quite regularly experienced in their day to day life – as looking into mirrors is likely to be a particularly familiar activity.

It would be surprising if personal meanings of reflection associated with these day to day activities were to match exactly the mathematical definitions associated with institutional knowledge of reflection. What is important about a mirror reflection is what is seen (the image) because this provides a way of viewing what is reflected (the pre-image). The position of the image relative to the mirror is not likely to be a focus of attention. In contrast, what are emphasised in mathematical definitions are the geometrical relationships between pre-image points, the axis and image points. How to bring these two worlds together was one motivation for this research.

The study involves the design and comparison of learning systems built to mediate between students' personal knowledge of reflection and the institutional mathematical knowledge they are supposed to learn. Two aspects of mediation were selected for investigation: the ways in which different instructional approaches might serve as connecting links between what students know and how they are supposed to know it; and the mediational role of computational tools with which students can simultaneous act with, see and express mathematical properties as general relationships or specific cases.

To focus on teacher mediations, two instructional approaches were developed and, to investigate tool mediation, two sets of computational tools were designed. These were combined, along with a set of paper-and-pencil activities and computer-based tasks, into four learning systems designed for use with groups of six girls aged between 12 and 13 years. Learning systems were compared by examining if and how knowledge about reflection (in a line and in two dimensions) came to be expressed in different ways according to the resources available within the systems.

Developing an instructional approach involves first making some assumptions about the relationships between students' personal mathematical knowledge and culturally embedded mathematical knowledge. Chapter 2 introduces two theoretical perspectives on how mathematical ideas emerge as learners interact in systems. Both perspectives stress the importance of constructive activity in the building of internal cognitive resources, but identify different approaches to mediate the flow between personal and institutional knowledge within learning systems, which were used as the basis for the instructional approaches, termed *filling-outwards* (FO) and *filling-inwards* (FI), used in this study.

Chapter 3 focuses particularly on tool mediation and presents a model of mathematics learning in which learners interact with developing tools and symbol-systems in evolving learning systems. The practice of mathematics is associated with formalisation, which involves the making explicit of mathematical objects, properties and relations through their expression in a mathematical symbol-system. The chapter also introduces the idea of a *microworld*, a special arena aiming to support a formalisation process that does not require the detachment of formalisms from the phenomena they express. It goes on to consider research specifically related to the learning of transformation geometry and particularly the isometry reflection.

Chapter 4 explains the methods by which the issues raised in the preceding chapters became operationalised and investigated. The empirical work was divided between two phases: the design phase, in which the main emphasis was directed towards creating tools, tasks and teaching interventions that would enable learners to extend their knowledge of reflection; and the comparison phase, which concentrated on analysing how learners' meaning-making activities were contingent upon the mediational means with which the systems were endowed.

Chapter 5 describes the research activities of the design phase. These included probing students' knowledge of reflection as expressed in paper-and-pencil settings in order to develop a picture of the knowledge students could be expected to bring to the system; the building of accessible and relevant microworld kernels and deciding upon the nature of teacher interventions before, after and during learners' microworld activity.

Chapter 6 presents the comparison phase. It describes how four groups of students were selected on the basis of their responses to a paper-and-pencil test and goes on to analyse the evolution of the four learning systems from a variety of perspectives: evolutions in students' expression of mathematical meanings for reflection and its objects; evolution of the microworlds tools; and evolution of the instructional approaches. In this way, it examines how the different forms of mediations shaped and were shaped by the interactions of system participants.

Finally, Chapter 7 looks back over the discussions in the preceding chapters and presents the main findings of the thesis.

Chapter 2

The construction of culturally informed knowledge

> "...any theory about children's growing understanding of mathematics has to account for at least two factors. One is the learning of invariants: the second is the acquisition of cultural tools..."
>
> (Nunes and Bryant, 1997; p.48)

It can hardly be controversial to claim that we develop and that we learn by interacting within the various biological, social, cultural and even virtual systems that make up the world as we experience it. Or that we evolve as individuals and we evolve as societies. The meanings individuals evolve for the knowledge culturally labelled as mathematics depend upon the ways they come in to contact with it as well as upon their own individual resources – physical, visual, auditory and mental. At the same time, mathematics is an ever-evolving practice, which has developed (and is developing) over time as a result of activities of individuals with the cultural heritage laid down before them. It could be argued, as, for example, Sutherland and Balacheff (1999) have, that it is therefore useful to recognise two types of mathematical knowledge:

> "(i) knowledge as shared intellectual constructs which allow co-operative learning, communication and mutual control, and (ii) knowledge as personal intellectual constructs"
>
> (p.2)

In the mathematics teaching that goes on in schools, the *intended knowledge* is the former – aspects of the socio-cultural artefact (that changes over time) known as mathematics. To be perceived as mathematically literate, learners need to construct mathematical meanings that are not only efficient in solving problems, but are also coherent with those socially recognised (Balacheff, 1991). The art of teaching is to nurture and respect learners' personal knowledge at the same time as supporting them in developing and communicating what they know in ways that emphasise its relationship with the intended, institutionalised knowledge.

The question of the relationship between personal, social and cultural aspects of knowledge and its development and learning has been addressed from a variety of points of view in a multitude of different research contexts – education, philosophy, psychology, sociology, anthropology to name just some. Within the mathematics education literature we can point to two broad theoretical "camps". In the first, the impetus for the development of logico-mathematical reasoning comes from the individual and involves active construction of personal knowledge, progressing from individual relationships with material objects and private symbolism to social practices and collective symbolism. In the second, the direction is reversed and it is the social rather than the individual plane that drives learning and development.

While somewhat of an oversimplification, the first view can be used to group together theories that have collectively become *constructivism* and can be described as having a connection to, if not a direct hereditary path from, the work of Piaget. The second trend describes perspectives linked to *sociocultural* theories of learning inspired in a large measure by Vygotsky. As Confrey (1994) suggests, research in mathematics education continues to draw heavily from the work of these two scholars. In the following sections a sketch of each theory will be outlined briefly to give an indication of the key concepts that can be found in contemporary theories of mathematics education. By reference to these introductions, aspects of the design of instructional activities and practices associated with these theories will be considered.

2.1 Constructivism and the Legacy of Piaget

In this section, five aspects of constructivism are briefly discussed, from their roots in Piagetian psychology to their transposition to the mathematics education context: construction of personal knowledge; social interaction; hierarchical levels of development; role of the teacher; and language and tools.

2.1.1 Construction of personal knowledge

For Piaget, every act of intelligence is characterised by an equilibrium between the assimilation of events, objects or situations into existing ways of thinking and the accommodation of existing mental structures to incorporate new aspects of the external environment (Piaget, 1951). Human beings are viewed as self-regulating, self-organising, living systems (Steffe, 1996) whose conceptual operative structures come about as they strive to maintain an equilibrium in the face of the endless movement in a system driven by biological maturation, sensory experiences, and social interaction. The individual actively manipulates or transforms his or her world in ways that have already proven *viable*.

The concept of viability comes from von Glasersfeld. He maintains that, in cognising activity, the individual does not aim to construct a mental copy of some outer reality or "truth"; rather thinking is instrumental. Satisfaction results from the successful accomplishment of the task or goal. Hence knowledge serves to organise experience. In this sense, cognition is entirely subjective and "neither does nor can concern anything except the experiential world of the knower" (von Glasersfeld, 1996; p.308).

The equilibration process is frequently described in terms of scheme theory (Piaget 1971; Dubinsky, 1991; Confrey 1994). Using the description in Confrey (1994), knowledge comes about as a result of a three-stage recursive process involving a "problematic, an action and a reflection" (p.4). After managing to operate successfully in a way that resolves a perturbation, a learner may consciously attend to the effective pattern of actions (operations) which brought about this resolution, and store it as a routine for future action. Where the learner becomes aware of the schemes and is able to use them consciously for prediction and explanation, they can be said to have engaged in *reflective abstraction*. So, from this perspective, logico-mathematical knowledge derives from reflective abstraction involving the isolation by an individual of the properties and relations of their own operations. This process can be contrasted with that of the building of physical knowledge through empirical abstraction, which, rather than attention to operations, involves the isolation of properties and relations of material objects (Vergnaud, 1997; p.7).

2.1.2 Social interaction

From the Piagetian perspective too, comes the recognition that social interaction has an important role in the development of thinking. This was given particular precedence in the classic studies in the 1980's of the phenomenon of socio-cognitive conflict (e.g. Perret-Clermont, 1980; Mugny, Perret-Clermont & Doise, 1981). In situations in which peers interacted together in learning tasks, the confrontation of difference is disequilibriating and each participant, perturbed by the conflict, attempts to restructure his or her understanding in the light of an alternative view (Kruger, 1993). Through such encounters, the learner can become more consciously aware of his or her own operational practices and the corresponding practices of others. Balacheff (1991) points to Piaget's contention that learners become aware of contradictions only when they possess the necessary cognitive resources to overcome them. If they are not *ready*, they will experience no disequilbrating effects from alternative viewpoints. Balacheff argues that this contention is too strong and suggests it is critical only that a contradiction be seen to provoke the initiation of a, perhaps long, process of overcoming. From this it can be concluded that the experience of socio-cognitive conflict can have the effect of sparking the learner in embarking on the essentially private cognitive process of reflective abstraction.

2.1.3 Hierarchical theories of mathematics learning

Probably the most well-known facet of Piaget's work amongst those concerned with mathematics education is his stage theory (Bidell, 1988; von Glasersfeld, 1994). Two important comments can be made about this theory. First, it is based on the recognition that a child's way of thinking is significantly different to an adult's. Second, it assumes that development from one to the other universally follows successive hierarchical stages from sensori-motor, through concrete operations, to abstract thought (Confrey and Costa, 1996). Both these have had important influences on research work within the mathematics education community. The first idea spurred considerable research efforts in identifying precisely how learners' approaches and perspectives differed from the accepted (and intended) orthodox mathematical practices (see, for example, Hart, 1981).

The second can be linked to the development of various hierarchical theories of mathematical education. Confrey and Costa (1996) point to a collection of contemporary theories, that they call *theories of reification* (p.140). They argue that these theories express a strong Piagetian influence by equating progress with the construction of increasingly "abstract mathematical objects" by successively leaving behind concrete operational referents. Central to work in this paradigm has been the discussion of the object-process duality of mathematical thinking and, in particular, the developmental relationship underlying this duality (see, e.g. Dubinsky, 1991; 1997; Sfard, 1991; Sfard & Linchevski, 1994; Gray & Tall, 1994). Sfard and Linchevski (1994) describe the growth of mathematical thinking (focussing on the context of algebra) as "a sequence of ever more advanced transitions from the operational to structural outlook" (p.191). In a similar vein, Dubinsky (1997) describes his view of the development of mathematical knowledge:

> "Mathematical knowledge begins with actions that are mental or physical transformations of objects. These actions are interiorized to processes. That means an internal mental operation is constructed to do the same job as did the more external action. Finally when it is necessary to apply an action to what has become an object, the process is encapsulated to become a mental object."
>
> (Dubinsky, 1997; p.71-72)

Although Confrey and Costa stress the endorsement in these theories of reifications of the classical Piagetian hierarchy, they can be even more closely related to a second less known epistemological hierarchy described in Piaget's last book, which he wrote with Garcia (Piaget & Garcia, 1989). This framework proposed that the development of a major mathematical idea passes through three stages:

- the intra-level where actions are performed with the objects of consideration and attention is directed to the objects themselves;
- the *inter-level* where the focus is between objects, on the relationships and transformations by which objects are associated;

- the *trans-level* which is characterised by the construction of an underlying structure into which the relations of the previous level can be coherently organised.

Each stage operates upon the objects from the preceding stage, transforming them so they can become participants in the next level. This continues, as in the reification theories, following an ordered and iterative *process of abstraction*. The structures emerging from trans-level actions can be used as intra-objects in a new round of the process. The intra-inter-trans triad, like theses-antitheses-syntheses in classical dialectics, has its basis, according to Piaget and Garcia, in the "role of disequilibria and re-equibriliations" towards higher levels of thought (p.134). This new triad, however, is argued to be more flexible than its classical counterpart. For the authors, it is a mechanism so general and constantly repeatable that it can be found in the transition from one level to the next in epistemological development irrespective of the absolute height of the levels. To support the epistemological sequencing of these three stages, Piaget and Garcia claim that they can be located in history as well as in the individual. For example, with respect to the development of geometry, they write:

> "Geometry begins with Euclid – with a period during which the object of study is geometrical properties of figures and solids seen as internal relations between elements of figures and solids. No consideration is given to *space* as such, or consequently, to transformations of these figures within a space that contains them all. We shall call this period *"intrafigural"* – an expression already used in developmental psychology to account for the development of geometrical concepts in the child.
>
> The following period is characterized by efforts to find relationships between the figures. This manifests itself specifically in the search for transformations relating figures according to various forms of correspondence. However these transformations are not yet subordinated to structured sets. This is the period where projective geometry predominates. We shall call this period *"interfigural"*.
>
> Following next is a third period, which we shall call *"transfigural"*. It is characterized by the predominance of structures. The work most characteristic of this period is the *Erlangen Program* of Felix Klein.
>
> (Piaget and Garcia, 1989; p.109)

This description of epistemological differences between classical Euclidean geometry and transformation geometry is particularly relevant to this study and will form an important focus of the analysis of student meaning-making.

2.1.4 Role of the teacher

Dubinsky (1994) suggests that Piaget was not especially concerned with the pedagogic implications of his ideas. Mathematics educators, however, have imported aspects of his theorising in their attempts to characterise situations conducive to learning mathematics. Unlike Piaget, in their applications, constructivist mathematics educators tend to assign an important role to the teacher. The theory of didactical situations (Brousseau, 1997) serves as a useful example. Piagetian cycles of problematic, action and reflection can be associated with the adidactical situations outlined in this theory: problematics can be engineered with the intention of provoking a conceptual rupture, an activity in which learners are confronted with a situation which cannot be resolved using their existing meanings. The rupture alone is, of course, not enough to engender the reconceptualisation and the situation must be structured so that it affords the means (possible actions and resulting feedback) for successful resolution.

Following situations of action, the different meanings suggested by them must become the focus of attention. In didactical situations, moments of formulation (establishing a language through which learners can communicate their actions, and models of actions, to others) and validation (providing reasons to accept a true conjecture or reject a false one) allow learners to reflect on their activity and to consolidate new meanings.

This consolidation must also pass through an institutionalisation phase in which "some external person must look at her [the learner's] activities and identify those which are interesting and have a cultural status" (Brousseau, 1997, p.44).

From a constructivist perspective, and consistent with a Piagetian approach, the learner's cognising activities during the problem-solving activities could be interpreted in terms of the scheme theory outlined above. The learner experiences a perturbation, which disturbs internal equilibrium. To resolve the conflict, a new meaning and the production of a new scheme are required. The situation has been fortuitously structured to guide the construction of such a new scheme; that is, to encourage reflective abstraction involving the operations underlying it. In contrast to the Piagetian model, the theory of didactical situations necessitates the presence of the teacher as the engineer behind the learning. The teacher makes sure the intended knowledge is not forgotten. Hence, it could be argued that the learners' activities result in them forming the necessary internal resources to assimilate, or perhaps even accommodate to, the historical and cultural counterparts and ways of representing the knowledge emphasised by the teacher during institutionalisation.

2.1.5 Conceptual structures, language and other tools

Common to the various constructivist perspectives introduced thus far, is the association of mathematics learning with some process of abstraction, whereby actions and processes become recognised as objects in themselves (Kent, 1998). Dubinsky's view most closely corresponds to the Piagetian tradition since he argues it represents *the* means of transition between ways of thinking. For him, reflective abstraction involves the encapsulation of all possible processes involving the objects of concern into a single totality or structure. Accordingly, before learners are able to use notations (and presumably other cultural artefacts) they need to have built the conceptual structures "fitted" to these artefacts. That is, the usefulness of the symbol system necessitates the pre-existence of the concept.

> "Once it [encapsulation] has been accomplished, however, a notational scheme can be developed and connected to the concept by the relatively simple act of associating a syntactically governed set of symbols with a mental object that an individual has already constructed."
>
> (Dubinsky, 1994; p.169)

Dubinsky focuses above on the learner in isolation, but, according to Sierspinska (1998), the idea that conceptual structures determine language use dominates the constructivists' position on language as an activity of communication that does not change the course of development, but only provides evidence of it:

> "According to Piaget, when the child actually starts to discuss his or her ideas with others, it is because his or her thinking is more geared toward comparing different points of view, and not the other way round."
>
> (Sierspinska, 1998; p.37)

According to Nunes and Bryant (1997) the argument that social phenomenon and cultural artefacts can have an impact on mental life only after the groundwork has been laid by individual cognitive development is becoming less and less tenable, with growing evidence that the activity of the same individual varies widely across different practices. While in one context (usually a familiar, everyday one) reasoning of an individual is indicative of an awareness of the mathematical invariants necessary to solve a problem, when the same mathematical invariants are encountered in a practice associated with a different semiotic discourse (usually a school-related practice), a much lower level of success is achieved. Actually, this finding alone is not enough to reject the Piagetian story. It could be said that the child has the necessary conceptual structures but has not yet incorporated the particularities of the means of communication used in the second practice.

A more direct challenge to the constructivist argument that conceptual construction comes before the expression of invariants in particular notations, is proffered by Nunes (1997) when she claims that different systems of signs enable and constrain possible problem-solving strategies. In this way, sign systems not only communicate but also shape the meanings associated with the activities within any practice.

Kaput (1994) has attempted to incorporate such an analysis of the role of notation in development into the constructivist view. While he appears to accept the intra, inter, trans hierarchy proposed by Piaget and Garcia (*§2.1.3*), for him it is the symbol system that facilitates qualitative changes in the thinking process and provides the means for transition upwards towards increasingly sophisticated levels of mathematical reasoning. This questions the assumption that thought precedes language and supports the idea that cultural tool and conception are reflexively related.

This view appears to be gathering momentum among constructivist theorists, who are beginning to respond to criticisms that semiotic considerations typically play little if any role in their accounts of mathematical development or in the terminology of assimilation, accommodation, schemes, equilibrium and reflective abstraction (see, for example, Sfard, 2000; Cobb, Gravemeijer, Yackel, McClain & Whitenack, 1997). Such analysis involves acknowledging the mediating role of languages and moves beyond the Piagetian perspective to the work of Vygotsky and the development of sociocultural theories of learning.

2.2 Vygotsky and sociocultural theories of development

According to Glassman (1994), Vygotsky shares with Piaget the same basic beliefs about development. The major agreement between them is that there are two parts of development, two lines that continuously interact with each other in the development of behaviour. There is the natural line of ontogenetic development and the dynamic impact of social/cultural development, an *active child* and an *active environment.* (p.307).

A major disagreement can be found in that, whereas Piaget postulated a co-equal relationship between individual and environment (e.g. Piaget, 1971 p.114; Tudge & Winterhoff, 1993), Vygotsky assigns analytic primacy to the social and cultural rather than the individual:

> "Every function in the child's cultural development appears twice: first, on the social level, and later, on the individual level; first between people (*interpsychological*), and then *inside* the child (*intrapsychological*). This applies equally to voluntary attention, to logical memory, and to the formation of concepts. All higher functions originate as actual relations between human individuals."
>
> (Vygotsky, 1978; p57)

2.2.1 The role of culture

Cole and Wertsch (1996) argue that the major difference between the two theories is expressed in relationship to the *role of culture* ("the accumulated products of prior generations") and, especially, the role of mediation of action through cultural artefacts on the development of mind. A central tenet of sociocultural theory is that human beings have a special mental quality which involves the need and ability to make use of artefacts – be they physical, symbolic or cognitive – to mediate their activities and to encourage the appropriation of these forms of mediation by subsequent generations (Cole and Wertsch 1996). Artefacts are created at a particular moment in the historical trajectory of a culture, as a response to the demands of a particular practice. These artefacts modify the activities of those using them and can also, in their turn, be modified in use.

2.2.2 Material and Psychological tools

Material tools serve as mediators between the human body and an object of action, they are externally orientated, affording to humankind the means to control, or at least purposefully manipulate, nature. The concept of a psychological tool arose by analogy with material tools. These too are artificial human inventions. In their external form, they are symbolic artefacts (signs, symbols, languages, formulae, graphic devices). Vygotsky argued that such symbol systems restructure mental activity in a manner similar to the way that material tools restructure physical activity. This allowed him to claim that basic psychological processes (memory, attention, reasoning) are universal human processes, but that their functional organisation varies, depending on the nature of the psychological tools that are available to the society and, via society, to the individual.

> "...by being included in the process of behavior, the psychological tool alters the entire flow and structure of the mental functions. It does this by determining the structure of a new instrumental act, just as a technical tool alters the process of natural adaptation by determining the forms of labor operation".
>
> (Vygotsky 1981, p.137)

All higher mental functions according to Vygotsky are culturally mediated. Or, to put it another way, the mental functioning that characterises human thought inherently involves the use of mediational means. A mediated mental function involves an indirect action on the world, which incorporates and transforms the natural, basic mental processes, extending their range and mode of functioning. The inclusion of the tool in activity hence alters the course both of the activity and of all the mental processes that enter into the instrumental act. In this way, tools do not only facilitate mental processes, they transform, they re-organise and they shape them (Vygotsky, 1981; p.139).

Various contemporary descriptions of the development of sociocultural theories have documented a serious ideological difference between Vygotsky and other Soviet theorists, centering on the distinction between material and psychological tools (for details of this rift see, for example, Kozulin, 1990; 1998; Cobb, Perlwitz and Underwood, 1996). A similar criticism re-emerges in the mathematics education community: Confrey (1995a) points to a privileging of abstract sign use over functional practical intelligence that can be found in Vygotsky's original work, which she is worried may lead to the neglect or devaluation of concrete activity. She argues also that

> "...in directly replacing the tools of labor with the psychological tools of signs, he virtually severed the connections between signs and the underlying tools, actions and operations that produced them and placed his emphases solely on the movement among signs"
>
> (Confrey 1995a; p.207)

In this vein, Noss and Hoyles (1996) argue that Vygotsky tended to overshadow informal, practical and enactive concerns in favour of formal and theoretical aspects (p.42-43). As a possible response to this criticism, the next section considers Vygotsky's view of concepts and, in particular, focuses on Vygotsky's distinction between spontaneous and scientific concepts, in which he does seem to have paid some attention to connecting, through the mediations permitted by tool-use, theoretical factors with their empirical counterparts.

2.2.3 Development of concepts

The use of concepts occurs within activities. At any particular moment, the less experienced actor (let's say the learner) engages in forms of thinking that depend on the resources available to mediate the activity and on the collaboration with, or rather the guidance of, other more well-versed actors (a teacher, for example). During activity, the learner and teacher enter into a dialogue in which, at times, the learner may display reasoning patterns before having gained conscious awareness of them. The teacher can exploit such displays to provide external signs of the concept in question in its culturally sanctioned form, reinterpreting and rephrasing the learner's contributions. To the extent that the learner is able to participate in the dialogue, he or she experiences possibilities of becoming more conscious of the underlying structure of his or her thinking, gradually internalising more advanced levels of performance. In this way, a learner can accomplish much more in collective activity than alone. Or, as Vygotsky had it, there is a:

> "...distance between the actual development level as determined by individual problem solving activity and the level of potential development as determined through problem solving under adult-guidance or in collaboration with more capable peers"
>
> (Vygotsky, 1978; p.86)

In this very well-known quote, Vygotsky introduces the notion of the *zone of proximal development* or zpd. The zone of proximal development is not a property of the individual – some innate capacity, waiting in an inert form to be awakened. Rather it is an intersubjective zone created in activity. Hence, it represents the conceptual site of learning, for the internalisation of functions from the social plane to the individual. It can be understood as the distance between the cultural or intended knowledge and the personal knowledge at play in the activity setting. (Davydov & Markova, 1983; Hedegaard, 1988; Lave, 1996). Vygotsky described schooling as an activity especially associated with the creation of zpds and this led to the distinction between *scientific concepts* and *spontaneous concepts.*

Scientific concepts seem to have initially been described as those originating from the rather special kinds of interaction particular to schools and are characterised by their systematic, logical organisation. They are described as dynamic changeable structures (Vygotsky, 1962). Spontaneous concepts, on the other hand, are described as those emerging from the learner's everyday interactions, mediated, of course, by cultural and social artefacts (Kozulin, 1990).

Davydov (1988) felt that Vygotsky's apparent limiting of scientific concepts to those introduced in school resulted in a rather procedural definition, which needed expanding from an epistemological angle. He stressed the theoretical content of scientific concepts and he contrasted this with the empirical nature of spontaneous concepts. Wardekker (1998), on the other hand, suggests the procedural distinction might be more consistent with the overall Vygotskian framework, since accepting the epistemological view would imply that scientific concepts are universal ideals, transcending history and conveying "truth". Because the registers associated with generalisable, logical reasoning are cultural constructs, scientific concepts could be considered simply as those very concepts that have been developed and that are in use in scientific practices. That is, scientific concepts can be thought of as those which result from social demand (Glassman, 1994). As such, scientific concepts represent the current models used to understand the "essence" of a phenomenon, the role it plays in human activities as they occur in their cultural and historical contexts (Il'enkov, 1977). Not objective truths then, but a general "truth" about the connection between object and activity. From this view concepts:

> "...are models: representations to ourselves of what we do, and of what we hope for. The model is not, therefore, simply a reflection or copy of some state of affairs, but beyond this a putative mode of action, a representation of prospective practice, or of acquired modes of action."
>
> (Wartofsky, 1979 xv)

Under this definition and, taking into account the previous description of activity in the zpd, scientific concepts appear to bear some resemblance to the reflective abstractions described by the constructivists, although there is still a rather fundamental difference. Reflective abstractions, in the constructivist sense, work upwards, moving thinking structures towards more general and systematic organisation. To describe the reciprocal relationship Vygotsky sees between scientific and spontaneous concepts, he also employs a metaphor of growth:

> "In working its slow way upwards, an everyday concept clears the path for a scientific concept in its downward development. It creates a series of structures necessary for the evolution of a concept's more primitive, elementary aspects, which gives it body and vitality. Scientific concepts, in turn, supply structures for the inward development of the child's spontaneous concepts toward consciousness and deliberate use."
>
> (Vygotsky, 1986; p.194 quoted in Kozulin, 1998; p.49)

In their appropriation as personal knowledge, scientific concepts grow downwards or inwardly to meet and intertwine with the upwardly growing spontaneous ones, fusing into a form of thinking that is all at once personal, social and cultural, in the sense that general classes and transformations characteristic of particular societies become "bound into the networks works of personal experiences and perceptions unique to each person" (Renshaw , 1996; p.63).

The use of the downwards metaphor to describe the learning of scientific concepts illustrates another significant difference between Piaget and Vygotsky. Vygotsky is suggesting that scientific concepts develop following a process in which the learner is presented with concepts of a general applicability which are valued in the culture (Saxe and Posner, 1983) and can be connected with experiences of a more particular and personal nature.

2.3 Upwards and downwards or *inwards* and *outwards*?

Until now, *abstraction* has been the major mechanism associated with the transition between qualitatively different modes of reasoning and there is a tendency in both the constructivist and sociocultural literature to privilege abstract over concrete reasoning. This has come under increasing criticism in recent years. For example, Confrey argues that:

> "Allowing mathematics to continue to require students to disengage from their personal sources of experience and to learn a system of rituals that makes little sense to them but which will admit them to the ranks of the elite is one of the most effective ways of maintaining this oppression."
>
> (Confrey 1995b; p.41)

She is by no means alone in this charge of elitism; for example, from rather different perspectives, Noss and Hoyles (1996), Walkerdine, (1988; 1997) and Turkle and Papert, (1991) all make similar charges. The description of the two-way relationship between scientific and spontaneous – although perhaps not the intention of the Russian psychologists at the time it was defined (see Lerman, 1998) – permits the suggestion of an alternative process that Wilensky (1991) has termed *concretion.* Using Wilensky's definition, concreteness is no longer a property of an object or thing, but "*a property of a person's relationships to an object*" (Wilensky, 1991; p.198 emphasis in original). This leads him to suggest "the actual process of knowledge development moves from the abstract to the concrete" (p.201). Like the reification theorists described in *§2.1.3*, Wilensky seems to be saying that knowledge development is characterised by movement between concrete and abstract, though for him the development proceeds in the opposite direction. Building from Wilensky's view, Noss and Hoyles (1996) offer a definition of concrete as "well connected to what the learner already knows" (Noss & Hoyles, 1996; p.73) and see abstraction as the means of adding new connections:

> "Abstracting can be seen as a way of layering meanings on each other, connecting between different ways of knowing and seeing, rather than replacing one kind of meaning with another."
>
> (Hoyles & Noss, 1996; p.73)

Mason's (1989) view of abstraction of a "delicate shift of attention" (p.2) in which learners move between seeing and using expressions as processes or objects also stresses the layering rather than letting go of meaning. The idea of layers of meanings suggests the use of less hierarchical metaphors to describe the relationships between different forms of knowledge. Instead of upwards and downwards, learning can be described using the metaphors of *inwards* and *outwards.* Constructivists who focus on the construction of knowledge by learners' progressive abstractions during attempts to organise their experienced world emphasise the outwards layering of meanings towards the intended knowledge. Socioculturalists who argue that development involves an internalising from an outer to an inner plane emphasise layering inwards. Neither constructivist nor sociocultural schools can be considered as inclusively inwards or outwards: the environment acts back on the learner in the former and, within the individual at least, there is some outwards layering in the latter. Figure 2.1 attempts to characterise schematically these differences.

Figure 2.1: Knowledge development from a social cultural perspective

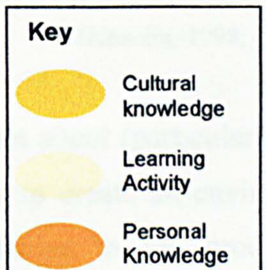

Figure 2.2: Knowledge development from a constructivist perspective

It is important to stress that theories of learning do not in themselves define instructional approach. Nonetheless, the differences in the flow of development, as presented in Figures 2.1 and 2.2, suggest that strategies aiming to maximise connections between personal and cultural knowledge might also differ. This raises the matter of the design of instructional materials and practices which will form the focus of the next section.

2.4 Design of instructional activities and practices

> "The central difference between the Piagetian and Vygotskian approaches lies in their respective interpretation of who is the active subject the learning process. For Piaget it is the individual child, who through active interaction with the physical and social environment, enhances his own cognitive schemata. For Vygotsky the learning process has a sociocultural character from its very beginning. The true subject of learning is an integrative whole that includes the child, the adult, and the symbolic tool provided by the given society. If in the Piagetian system the child is presupposed as a true agent of his or her own learning, in the Vygotskian system the child becomes an independent agent at the end of the formal learning experience. These two opposite approaches have found their realisation in classroom practices."
>
> (Kozulin, 1998; p.3)

In co-ordinating their specific goals with what is known about (particular) students' thinking, the designer of instructional material strives to create an environment, a system, in which students can act in ways conducive to the production of mathematical knowledge (Greeno, 1991; Cobb, 1997). This is still a very global statement of intent.

Gravemeijer (1997a) stresses that much of the development work in mathematics education has been guided by philosophies rather than instructional theories (he, for example, works within Freudenthal's philosophy concerning realistic mathematics education). Philosophies provide powerful heuristics, but not procedural guidelines. Hence, instructional sequences generally fit the adopted educational philosophy. Developmental research in instructional design, according to Gravemeijer, consists of a cyclical process of thought experiment and teaching experiment. Instructional activities are designed on the basis of *a priori* analysis in which hypothesised learning trajectories (Simon, 1995; p.133) connected to particular mathematical content are envisioned. The activities are then enacted in the classroom and an analysis of what happens is used to guide the next instructional activity.

Gravemeijer (1997b) points to two contrasting heuristics that can be used to structure the overall presentation of instructional approaches and which he distinguishes using a metaphor borrowed from information-processing theories. In the *top-down* approaches, "formal crystallised expert mathematical knowledge is taken as a starting point for developing instructional activities" (p.316). This means that, in top-down models, prefabricated systems of mathematical knowledge represent the immediate goal for instruction and activities focus on connecting the informal and situated personal knowledge of the students with this given system. In *bottom-up* approaches, on the other hand, the initiative is with the students and the idea is to make it possible that all their mathematical knowledge emerges out of their own constructive efforts. Instead of presenting students with given models of general applicability and symbol systems, bottom-up approaches encourage students to come up with their own. In light of the discussion in the previous section, and the revisioning of the relationship between concrete and abstract in mathematics learning, the terminology of top-down and bottom-up no longer seems advisable. Rather than top-down, the first approach could be characterised as a *filling inwards* from intended to personal knowledge. Similarly the bottom-up approach becomes *filling outwards* from personal to intended knowledge.

In relation to sociocultural theories, where emphasis is on appropriation of cultural artefacts and priority is given to social and cultural processes when accounting for individual activity, it is perhaps not surprising that examples of filling-inwards approaches can be located. For example, Lave (1997) describes a curriculum of tailoring in which the outermost level is "learn to make garments" and the next level "learn to sew, then learn to cut them out" (p.33). Apprentices start with a clear notion of the general goal, then fill in the procedures and details which will make this possible. Another good example is the instructional theory of "ascending from the abstract to the concrete" developed by the research group led by Davydov in the former Soviet Union. This example will be examined below as a generic example of the filling-inwards approach.

Filling-outwards instructional approaches can be seen as particularly compatible with constructivist epistemologies, with learners, on the basis of existing conceptions and interactions in activity, encouraged to mathematise progressively from specific solutions to more general representations. For example, in the description of Brousseau's didactic situations in *§2.1.4*, learners begin with a problem specifically aimed to challenge their current ways of thinking and to provide resources to construct new ideas. Only after this, in the institutionalisation phases, is the general goal made more explicit. Another theoretical perspective that can be firmly associated with filling-outward is the emergent/socio-constructivist perspective, which is bringing together socio-constructivists from North America and Dutch researchers from the realistic mathematics education paradigm (see for example, Gravemeijer, 1997b; Cobb, Perlwitz and Underwood-Gregg, 1998; Cobb, 1997; Cobb, Gravemeijer, Yackel, McClain & Whitenack, 1997). This particular group has been chosen as the vehicle to characterise the filling-outwards style because they explicitly state that their view considers "individual thought and social and cultural processes ... to be reflexively related, with neither attributed absolute priority" (Cobb, Perlwitz and Underwood-Gregg, 1998; p.152). The authors describe their perspective as seeking to co-ordinate constructivist analyses of individual students' activity with analysis of the social and cultural setting in which it occurs. They thus aim to provide a constructivist interpretation that takes into account semiotic mediation.

2.4.1 A filling-inwards example: ascent from the abstract to the concrete

A central tenet of this approach, as described by Davydov, is that theoretical thinking moves from the general to the particular:

> "Succeeding in making the particular visible through the general is a characteristic feature of the kind of academic subject which awakens and develops the child's ability to think theoretically..."
>
> (Davydov, 1975b; p.204 quoted in Renshaw , 1996; p.66)

His approach was to attempt to isolate central abstract ideas of the mathematical concept under study, which, along with their semiotic means of representation, were presented to students. The idea was that this would enable students to internalise the essence of the idea in the mental plane and then deduce particular features of the idea by exploring its manifestation in familiar and meaningful contexts.

> "By registering in some referential form the primary general relationship that has been identified, schoolchildren thereby construct a substantive abstraction of the subject under study. Concerning their analysis of the curriculum material, they disclose the rule governed link between this primary relationship and its diverse manifestations, and thereby obtain a substantive generalisation of the subject under study...When school children begin to make use of the primary abstraction and the primary generalisation as a way of deducing and unifying the other abstractions, they turn the primary mental formation into a concept that registers the "kernel" of the academic subject."
>
> (Davydov, 1988 Part 2; p.22-23)

Davydov claims that this method mirrors the way in which people have historically created knowledge (p.21-22). He is highly critical of what he called the empirical theory of knowledge, in which generalisations are made as a result of an empirical process. This process involves the data-driven identification and labelling of common properties of a group of concrete objects or observable phenomena. For instance, in an empirical approach the notion of circle might be introduced by generalisation from round objects which have a common form, but which do not embody its theoretical definition. As a contrast, a theoretical approach might introduce a circle as a figure produced by rotated a line segment – a definition simultaneously: genetic, a procedure for engendering circles; universal, a procedure for all circles; and theoretical, requiring no prior knowledge of circles (Kozulin, 1990).

The focus of a particular teaching sequence never strays from the primary general "kernel" and, although the teacher demonstrates the mathematical voice to be echoed (see also Boero *et al.* 1997), this does not mean that students are expected simply to abandon any personal understanding and unquestionably imitate the teacher's actions. Renshaw (1996) describes various teaching strategies used to promote connections between spontaneous and the scientific counterparts. These include the use of *leading questions* in which students were encouraged to use more general terms:

> "The teacher began with the words that the children brought with them into the classroom, and she assisted them by leading questions to employ other words, more precise, more abstract and more general, that could be used in place of their everyday words."
>
> (Renshaw, 1996; p.68)

A second teaching strategy involves the *staging of mistakes* by the teacher. To draw attention to particular relationships the teacher might purposely enact an inconsistency, giving the students a sense of their own authority in validation activities. Third is a strategy termed *clashing*, in which learners are invited to compare different (and correct) representations of the same relations.

A criticism directed at Davydov's work is that the teacher and student are not placed in a reciprocal relationship. In the zone of proximal development, it is the teacher who is expert and the student novice. While the novice uses the expert's voice in order to appropriate it and hence to learn, the expert already knows. Learning on the part of the expert in the course of teaching episodes is not normally a feature of filling-inwards approaches. Steffe (1996) is most concerned that this may result in suppression of the personal concepts of the children in favour of the cultural concepts (p.84). A second potential problem relates to what has become known as the *didactical paradox* (see, for example, Brousseau, 1997): the more explicit the expert is about what the novices are to do, the easier it becomes for novices to (re)produce the required behaviour without attending to the necessary knowledge.

Renshaw (1996) expresses another concern, that the sudden qualitative shift in conceptual thinking, from everyday judgements to symbolic representations, reinforced by the design of Davydov's teaching experiments runs the risk that many students will develop little more than empty formalisms. Actually this goes hand in hand with the worry about lack of emphasis on students' personal approaches. In sociocultural terms, the problem can be put as follows: without any clear model of the students' concepts and operations, it is difficult to see how the teacher can be sure of creating a zone of proximal development. If the teaching is aimed above (or below) the limits of this zone then no learning will occur. Although learners may attempt to use the symbols introduced by the teacher, if they are unable to connect the formal description to meanings they have already appropriated, they may have little idea of what they are doing or why.

2.4.2 A filling-outwards example: the emergent approach

Whereas an instruction activity structured in a filling-inwards manner starts with the presentation of general mathematical notion, in filling-outwards approaches the starting points are situations that should be "experientially real" for the students. This does not necessarily imply contexts connected to real-life situations, it means any contexts that have personal meaning to the participants and with which they can immediately engage in mathematical ways (Gravemeijer, 1997b; Streefland, 1991).

The experiential starting points should be justifiable not only in terms of their connection to students' knowledge, but also serve as appropriate springboards to launch students into ever advanced mathematical practices.

> "...students' initially informal activity should constitute the basis from which they can abstract and develop increasingly sophisticated mathematical interpretations as they participate in classroom mathematical practices. At the same time the starting point situations should continue to function as paradigm cases that involve rich imagery and, thus, anchor students' increasingly mathematical activity."
>
> (Cobb, Gravemeijer, Yackel, McClain and Whitenack, 1997; p.160)

The endpoint in this filling-outwards approach can be considered as analogous to the endpoint point of the filling-inwards method: that is, the construction of mathematical meanings in which general mathematical ideas are grounded in specific activities. What is different is the way this connection is expected to be made. In Davydov's description, it goes from abstract to concrete; in the filling-outwards approaches, development involves progressive mathematisation in the reverse direction:

> "There is a gradual exchange of concrete, contextual meaning and form into the mathematical meaning and form."
>
> (Streefland, 1997; p.370)

Another critical difference between the approaches is in the use of models and symbol systems. In filling-inwards approaches, it is normal for the teacher to present general models and the means through which learners should symbolise them. The aim is that the cultural meaning inherent in appropriate use of these symbols will become connected to students' personal meanings. When filling-outwards approaches are adopted, ways of symbolising are not construed as the means of bringing students into contact with established cultural meanings. They are thought to support the emergence of mathematical meanings, the reinvention of cultural meaning by the participants themselves.

> "To be sure, the teacher's and the instructional developer's understanding of the mathematical practices institutionalized by wider society provides a sense of directionality to this process of emergence. However, the basic metaphor is that of *building toward* participation in these practices rather than bring students into contact with cultural meanings"
>
> (Cobb, Gravemeijer, Yackel, McClain & Whitenack, 1997; p.163, my emphasis)

Within this instructional approach, there is no talk of appropriation, of students echoing mathematical voices, instead following the *reinvention principle*, the idea is that students construct models for themselves and that these models serve as the basis for developing formal mathematical knowledge (Gravemeijer, 1997a; p.339). Gravemeijer (1997b) describes how the progressive mathematisation from situation to formal knowledge develops through four levels: the level of situations, where domain-specific strategies bounded by the context are used; a referential level in which strategies refer to, or are *models-of* situations sketched in given problems; a general level, where the focus is on mathematical strategies to describe, or *models-for*, the context; and finally the formal level characterised by activity consistent with conventional mathematical procedures and notations (p.30).

Just as with the filling-inwards approach, it is important to consider the teacher's role in mathematising activity. Classroom activities are organised so that students first engage – act – individually or in small groups with a task designed to support the emergence of a particular mathematical idea. Students' actions with respect to the task become the topic of subsequent conversations mediated by the teacher.

It is interesting to note that teachers' mediation strategies are not so different from those described in the previous section. Both involve the comparison of solutions (either correct or incorrect) and both involve attention to mathematical voices. Where they do differ is in the relative contributions of the teachers and the learners. In the filling-inwards approach, the teacher asks quite direct and leading questions designed to encourage the students to adopt the mathematical voice. In the filling-outwards approach, the teacher, or other learners, might also restate the learners' voice from their own perspective, but the *re-voicing* is a matter for negotiation not appropriation. Confrey (1998) introduces a dialectic of voice and perspective in which another's voice can be used as a means to rethink one's own perspective.

Comparison of problem representations is another teacher-mediated strategy in the filling-outwards approaches. In contrast to the filling-inwards approaches where the teacher controlled the representations to be compared, in the practices described by Cobb and his colleagues, it was the learners themselves who decided whether their problem-solving approach could be counted as different from those of others. In filling-outwards approaches, then, learners are encouraged to engage in processes of comparison of *matching* or *contrasting* different problem representations.

A worry mentioned in association with filling-inwards approaches was the possible suppression of diversity in learner expression. In classrooms that manage to stay faithful to the filling-outwards approach, suppression of diversity should not be an issue. The teacher, and the learners, attend to voices of others and consider the potential viability of all the different strategies and solutions. What might be of more concern is that, although learners create a multitude of models-of their activity, these may not easily form the basis for abstraction into models-for conventional mathematical procedures, notations or conceptualisations. In the teaching experiment described by Cobb *et al.* (1997), it seems oddly inconsistent that the teacher could introduce highly structured tool-use but that this was not true for notation systems. They went further to suggest that it could be quite counterproductive for a teacher to suggest a particular notation system if students could not immediately imagine an activity that could have given rise to this symbolising (p.213).

And yet, in the examples of the teacher's contributions in this study (and in others such as Cobb, Perlwitz & Underwood-Gregg, 1998), it appears that the teacher does actually introduce new symbol notations (a learner's statement "four rolls [of sweets] and three pieces" is transformed into 4r 3p by the teacher, for example). For the researchers this is interpreted as capitalising "on children's interpretations, solutions, and explanations when guiding the development of classroom mathematical practices" so that "the teacher can fulfil his or her obligations to the school and to wider society without steering of funnelling children to predetermined responses that he or she has in mind all along" (Cobb, Perlwitz & Underwood-Gregg, 1998; p.77). It is clear that, while this notation closely matches the students' verbalisation, it did not seem actually to come directly from the students. It is also true that students were working with the numeration system, undoubted the most commonly used mathematical system in and out of the classroom. Would the approach described above be possible, if the intended knowledge involved completely new symbol systems? These concerns suggest that the filling-outwards approach should legitimise the introduction of new symbolic representation where students' actions indicate that this can be done in a single link of the signification chain.

2.5 Summary and research issues

Constructivist and sociocultural perspectives provide alternatives theories by which to attempt to interpret learners' interactions with their experienced world and how these relate to the construction of mathematical meanings. In the preceding sections, some important constructs in both theories were presented, along with some of their points of agreement and departure. To highlight some important common points:

- both are based on genetic analysis, recognising the importance of understanding individual intellectual growth alongside the historical, cultural development of knowledge;

- both offer a view of the human mind that is embodied and socially-situated: that is, that cannot be separated from the physical and social world it inhabits, acts upon, transforms and is transformed by;

- both imply that interaction with the material world and with other actors in this world is fundamental for development, that learning is an active process of adaptation of learners to their environments;

- both suggest that experience is transformed by some sort of reflection process;

- both posit that mathematical development progresses through shifts in thinking of a qualitatively different nature rather than a quantitative accumulation.

Some important different between the theories in terms of mathematical learning have also been highlighted which can be summarised as follows:

- in sociocultural perspectives, the developmental flow proceeds in an inwards direction as individuals internalise knowledge from the cultural plane to the individual one, while the constructivist view is of a more reciprocal relation between society and individual, with personal knowledge moving outwards towards knowledge of world they are adapting to (and that they adapt);

- traditionally, constructivists concentrated mainly on the communicative roles of language and tools, whereas sociocultural ideology assigns to them a mediating role in transforming, re-organising and shaping mental functions.

Since, as has been argued, today's constructivists are tending to incorporate the idea of symbol and tool mediation into their researches, it is particularly with respect to the first difference that the two instructional approaches outlined in the previous section will be distinguished in this study.

The filling-outwards approach can be succinctly summarised in the words of Cobb *et al.*, (1998) who argued that its idea is to develop instructional activities which

> "make it possible for mathematically significant issues to *arise out of* children's own constructive efforts in the course of classroom social interaction"
>
> (p.74; my emphasis)

To summarise the filling-inwards approach, this definition only needs a slight modification and can be stated as to

> make it possible for mathematically significant issues to become *appropriated during* children's own constructive efforts in the course of classroom social interaction.

§2.3.1 and *§2.3.2* presented differences between the two approaches in terms of the starting points in instruction episodes (global structuring), in the role of the teacher during group interaction (local structuring) and in the introduction of symbolic representation means. Whether the last of these was really a difference when it came to classroom practice was questioned, as it appeared that the teacher in both approaches did actually introduce formalised expression into the classroom interaction. The real difference was in the way they were introduced. The filling-outward teacher attempted to encourage students to re-describe their mathematical activities, whereas the filling-inwards teacher provided the students with the means with which they were to describe their mathematical activities in the first place. Figures 2.3 and 2.4 presents the two instructional approaches in diagrammatic form.

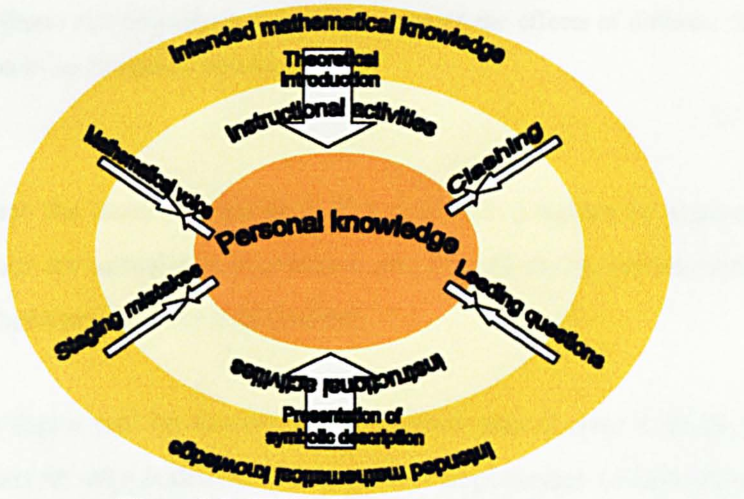

Figure 2.3: Filling-inwards approach

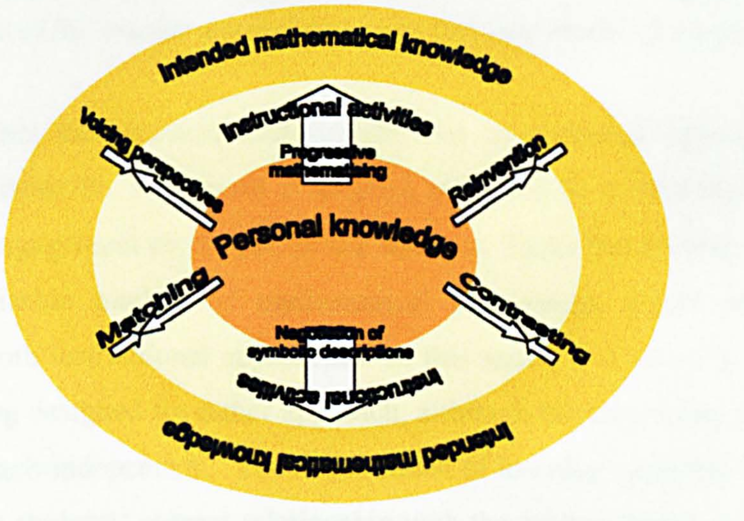

Figure 2.4: Filling-outwards approach

The differences in the descriptions of the two instructional approaches in theory and their actualisations in practice raise an important question. Since classroom are self-organising systems, to what extent is it possible, or even ethical, to maintain a particular approach strictly? The question was considered by van Luit and van de Rijt (1997). To describe instructional approaches under the filling-outwards umbrella they use the term *structuring instruction form* whereas filling-inwards approaches correspond with what they call *guiding instruction forms*. In a quantitative comparison of student learning outcomes, they found no evidence of difference between students who followed one or other of these approaches. However, they suggest that this may have been due to the fact that the teachers did not adhere religiously to the styles to which they had been assigned. They suggest:

> "Literature shows that little research has been done on the effects of different forms of instruction in mathematical education"
>
> (p.223)

And go on to stress the need for research of a qualitative nature to examine in detail how the approaches are actualised in practice and the effect the instruction has on the interrelationship between teacher and student.

One focus of this thesis will be to examine the development over a limited time-span of *learning systems* in which different instruction approaches (either filling-inwards or filling outwards) are operationalised. Or, in other words, *to systematically examine the mediating role of the teacher according to two different models for instruction.*

Like constructivism and socioculturalism, the two instructional approaches have many features in common. Whichever is adopted, tasks will be needed that call for an intensive active engagement on the part of the students. Tasks that involve students in building generalisable models of mathematical phenomena would seem to be appropriate for both instructional approaches. In this sense, it is possible to imagine similar tasks being designed for either approach, although the structuring of the tasks will not be approach-independent. The construction of learning materials involves an assessment of the students' current relationship with the mathematical content, along with a complementary epistemological analysis in order to address the appropriate and possible starting points and the hoped-for end points of any instructional intervention. So, problem situations need to be carefully chosen and so do the tools through which the students are able to act and to express their actions. The importance of particular forms of representation and tools, as well as their interrelationship with the mathematical content embedded in the task, will be considered in the next chapter.

Chapter 3

Mediation, expressive means and microworlds

> "It is clear that we need to focus on mediations, on that which stands between the individual and social learner, and the 'knowledge' which he or she is supposed to learn."
>
> (Noss & Hoyles, 1996; p.107)

There are a number of ways in which mediation is important when considering mathematics learning. The first relates to the structuring of learning systems that was considered in the previous chapter. In a learning system involving a teacher and students, the teacher has a mediating role. He or she is a means through which students access (or not) the intended knowledge. By choosing an instructional approach, emphasising either *filling-inwards* or *filling-outwards*, the teacher makes decisions about how their students will encounter a particular mathematical idea. As such they form a connecting link between students' knowledge and the intended mathematical knowledge.

In the design of an instructional setting, the teacher's role obviously involves more than the adoption of one or other instructional approach. It also involves making choices about tasks; it involves determining the media for instruction; and it involves decisions about the tools that will be included in the setting. Some general design criteria for tasks were also raised in the previous chapter. They should take into account the knowledge that students bring to the learning situation; they should evoke contexts that are meaningful to students; and they should involve students in constructive activity around mathematically significant issues. In brief, the tasks need to involve students in mobilisation of mathematics, and this involves some way of expressing their ideas.

This brings to the fore another form of mediation: the mediational role of the expressive means available within a learning system: the material and psychological tools participants use as they act.

This chapter will focus on this second form of mediation, first by looking in general at how mathematics symbol systems afford or constrain meaning and then by considering the means of expressing mathematical understandings and practices made possible by computer microworlds. Computer microworlds were chosen as the arenas for students' activity because of the particular demands involved in communicating with them. The process of acting in ways that the computer understands (that is can respond to), necessitates the microworld user to express explicitly the relationships behind this action. This has two upshots. First, it emphasises the constructionist surmise that learners build personal knowledge particularly well when they are engaged in the construction of public entities (see, Noss & Hoyles, 1996; p.61), an idea that provided the basis for the learning activities devised in this study. Second, as the microworld users communicate with the computational system, their thinking is projected into the vision of the observer (p.6).

Moving to the more specific concerns of the study, this chapter will introduce two microworlds offering different models of geometry and different tools for constructing and manipulating geometrical objects and relationships. Following this, discussion will turn to the learning of geometry in general and transformation geometry in particular by reviewing the existing body of research related to students' activities in this domain.

3.1 Mathematical symbol systems

When people interact, their ideas, positions, questions, solutions, dreams, needs and wants are expressed through the use of a common set of material and/or psychological tools. The extent to which any participant is able to express what she or he knows within any system of activity and the extent to which they are able to act in ways which lead to successful goal completion are related to their fluency with the mediational means of that system. This helps explain why performance is intimately bound up with setting.

In different settings, learners will be confronted with different ways of expressing their mathematical practices, and, as discussed in *§2.2.3*, the particularities of these expressive means can be expected to shape activity. Different mediational means enable (or afford) and constrain activity in different ways (Nunes, 1997; Roth, 1999). This means a mathematical problem becomes fundamentally changed when it is associated with different expressive means. For Noss (1997; p.289), expressing through a mathematical symbol-system is what defines the practice of mathematics. He would have it that mathematicians are involved in a discourse which has quite distinct rules, differing substantially from those operating elsewhere[1], and which, critically, must be explicit (p.304).

This making explicit of mathematical ideas can be defined as formalisation, that is *a description of a given situation which denotes its mathematical properties and the relationships between properties by reference to a particular theoretical, formal system.* Formalisations hence carry with them a degree of generality, in that any expression, at least potentially, signifies the existence of a class of objects and/or relationships rather than a unique event. Having defined formalisation, a definition of a formal system is also in order. Here is one furnished by Shaffer and Kaput (1999):

> "A formal system, then, is an arbitrary, but well-defined set of symbols and, most importantly, rules of transformation on those symbols. A critically important feature of such systems is their operative nature – the existence of internally coherent rules for transforming the allowable ('well-formed') symbols into other symbols".
>
> (p.107)

Understanding the ways mathematics is "done" in different settings involves understanding the different ways in which different formal systems mediate the construction and expression of mathematical ideas. Alongside the mediation role of instructional approach, this study therefore also concerns the mediation of learning role by the formal systems embodied in mathematical microworlds.

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[1] For other discussions of the differences between mathematical and non-mathematical discourses see, for example, Lemke (forthcoming) and Walkerdine (1988; 1997).

3.2 Computer-based microworlds for mathematics learning

In its beginning, the term microworld was associated with the domain of Artificial Intelligence[2], and with simplified "fairyland" models of limited domains of the experienced world (Minsky & Papert, 1970; p.70). Papert (1980) imagined accessible, evocative and engaging provinces of mathlands (p.125), or mathematical cultures, in which learners would become immersed, and from which they would emerge as more mathematically fluent. His conception was of computational objects, which would embed a mathematics that was not only formal but also "related to the self, the body, material and social objects, and activities" (Papert, 1992; p.xv).

> "The computer stands betwixt and between the world of formal systems and physical things; it has the ability to make the abstract concrete. In the simplest case, an object moving on a computer screen might be defined by the most formal of rules and so be like a construct in pure mathematics; but it is as the same time visible, almost tangible"
>
> (Turkle & Papert, 1991; p.162)

Balacheff and Kaput (1996) suggest that a unique feature of computer-based learning environments when compared to other types of learning material is their cognitive character (p.469). They represent rather unique interaction partners for students in that: they (re)act in response to communications – inputs – of users; their (re)actions are determined by their formal systems; and their responses – feedback – representable in various media are coherent with the rules of their systems. Only the first of these characteristics can be said of any other interaction partner or piece of educational equipment. These features are central to microworlds but do not define them. They can be applied to computer systems more generally.

The special quality of a microworld is that it provides access to a variety of formally defined objects and relationships *with which users interact in order to construct and manipulate new objects and relationships.* Microworlds are extensible. This

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[2] Hoyles (1993) overviews the etymology of the term "microworlds" and traces its genesis from the research corridors of universities to the school mathematics classroom.

characteristic rules out any "software restricted to an a priori set of self-contained allowable actions on a hermetically-sealed set of rules" (Noss & Hoyles, 1996; p.65).

At the core of any microworld for mathematical learning is a model of "a knowledge domain with epistemological significance" (Hoyles, 1993; p.2). The knowledge domain signals the *mathematical focus* intended by the designers. The initial model provided by the designers – the set of primitives, objects and operations on these objects they choose to make available – represents a *technical kernel* bringing together pedagogic, cognitive and epistemological considerations. As Thompson (1987) explains, the primitive tool-set of the microworld kernel, the designers' model as it were, captures the idea of a mathematical system. The full system is not given at the beginning but is realisable as the learner builds new objects and new relationships using the given primitives (p.85). So one part of microworld interaction necessitates formalisations, but this is not the whole story. The other part is the dynamic semantics microworlds provide for this formal system, the graphical displays that depict the actions of the objects of the microworlds. Action and formalisation can hence be simultaneously experienced.

As learners interact with microworlds they build their own computer-based models. These models are both personal and public entities, which reflect learners' thinking about the mathematical objects and relationships as they work on a particular activity at a particular moment. Following Papert, this takes one into the realm of the *constructionists'* view of learning (see Papert, 1991; Harel & Papert, 1991). In constructionism, a basically Piagetian perspective comprises the underlying rationale for the modelling phase:

> "Individual action is considered the motor for learning. Students use the software to achieve a goal and in the process they learn by co-ordinating and reflecting on the form of their interactions—by developing schemes."
>
> (Hoyles, 1995; p.203)

Emphasising individual action as the motor for learning may seem in conflict with the Vygotskian view of internalisation from the social to individual plane. It is not however inconsistent in the context of microworld activity. This is because the mediational means, or at least those built into the computational systems, have been designed precisely with idea of probable zone of proximal development in mind. The idea is that, working with these tools, learners can do things with a computer that would be impossible without it. The computer becomes synonymous with more-knowledgeable[3] other in this view. So the challenge becomes, as Hoyles (1993) put it, to build into the make-up of the microworld what she labels "evocative computational objects – *those which matter within the relevant knowledge domain and which matter to the learner*" (p.12, emphasis in original).

To make sense to learners, tools must connect with their points of view and must resonate with their ideas about how to express what they already know. The tools must also embody relevant mathematical knowledge. This means they may be a little different from material tools that help get things done.

> "Students must be able to use the tools to develop an explicit appreciation of the form of generalised relations, the relational invariants, while the functionality and semantics of these invariants — their meanings — are preserved and extended. To achieve this, the tools have to do "just enough" to illuminate structures and relationships without solving the task completely — a difficult balance to achieve yet one that is crucial if both student and mathematical meanings are to be respected."
>
> (Hoyles & Healy, 1997; p.28)

An important aspect of this study is to extend an understanding of how to characterise "just enough". This is an extremely complex endeavour. The tools play multiple roles: they mediate students' conceptions, determine the expression of mathematical ideas under investigation and, by serving as a means of communication, also form part of the mediating role of argumentation between peers. Moreover, they need to afford and constrain student activity in a way consistent with intended mathematical meaning.

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[3] More knowledgeable denotes only more expert in activity involving its formal system, in all other respects the user is of course far more knowledgeable than the machine.

3.2.1 Two microworld models of geometry

In Balacheff and Kaput's (1996) review of the impact of computer-based learning environments in mathematics education, two microworlds associated with geometry learning are given particular attention; Logo turtle geometry and the dynamic geometry system exemplified by the software Cabri-géomètre (Laborde, 1985). The models of geometry that underlie each of these respective microworlds are fundamentally different from each other.

Turtle geometry denotes a particular kind of geometry that has its origin in the study of artificial intelligence (Abelson & diSessa, 1980). It began life in the form of computer-controlled robots which moved around a two-dimensional surface in response to commands to go forwards or turn to the right. This provided a new way of regarding plane geometry figures as tracings made on a display screen describable by computer programs[4]. Geometrical designs can be produced on the screen by writing symbolic procedures in which the primitives of the systems are organised and then executed. As the procedures are run, the screen turtle(s) follows the encoded path, leaving traces as specified by the programmer. Any variability in the geometrical design is again under symbolic control, enabled by the inclusion of variables in the programming code. A figure drawn in such a way can be described as consisting of a set of intrinsic properties, depending solely on the figure in question and not requiring any external reference frame beyond the turtle itself.

In contrast to the turtle geometry associated with Logo microworlds, the Cabri-géomètre microworld provides a model of geometry much more closely based on traditional Euclidean geometry, with sets of primitive objects (e.g. points, lines segments, circumferences) and actions (geometrical constructions such as midpoints, perpendicular bisectors and the like). Instead of the use of a symbolic language,

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[4] From this starting point, Abelson and diSessa (1980) go on to consider turtle geometry methods of exploring topics such as perspective, vector operations, curved space, topology and general relativity.

figures can be constructed on the screen in physical ways controlled by the mouse:

> "Cabri-géomètre provides a 'real' model of the theoretical field of Euclidean geometry in which it is possible to handle in a physical sense the theoretical objects which appear as diagrams on the screen"
>
> (Laborde & Laborde, 1995; p.242)

Geometrical figures defined using the Cabri construction tools are different from their paper and pencil counterparts in that they are more than single static instances illustrating particular relationships. Certain elements of constructions can be dragged around the screen and, as an element is moved, the entire figure responds dynamically, preserving any relationships dependent on the specific construction processes underlying it.

So, while both Turtle Geometry and Cabri microworlds offer users a primitive set of tools with which they can interact to build mathematical objects and relationships, the models of geometry underlying these constructions and the ways they are expressed are very different. This raises an obvious question, supposing the mathematical focus and the task are the same: *are the processes by which knowledge develops in learning systems involving one or other of these microworlds radically different or essentially similar?* To date, this question has not been researched in any systematic way, though it has been addressed both by Balacheff and Sutherland (1994) and Hoyles (1995). Their conclusions were rather different.

Balacheff and Sutherland focus on the respective cognitive demands related to the same mathematical problem in the two microworlds. They hypothesise that the same problem has a different complexity in each environment and that learning that results from interactions with either software is likely to lead to the construction of quite different meanings. This was not a research study, so they present no evidence to back up either of the claims. They may have a strong intuitive resonance and may even be true if the students' goals coincide with the goals of the researcher. This is a big if. The problem is that, in practice, as Noss and Hoyles (1992) comment, although the microworld may be littered with the necessary means to produce the intended mathematical meanings, there is no guarantee that, given the freedom to construct their own solutions, learners will actually make use of these tools. They might impose a completely different interpretation on the activity than that intended by the designers. They call this the *play paradox.* (p.442). Allowing learners to play may encourage them to develop and give meaning to their own ideas, but it also gives them room to subvert the activity and ignore the intended knowledge.

It is in relation to the play paradox that Hoyles (1995) sees similarities between the two softwares. She suggests that, despite their differences, a common pattern of responses to microworlds based upon them can be observed and parallels drawn in terms of: students' use of the software as a scaffold[5] for developing understandings, or as a means to bypass mathematical analysis; and in the way that tools constrain users to act in ways that deviate from "official" mathematical practices, but at the same time throw light on them.

Like Balacheff and Sutherland, Hoyles' observations are derived from reflecting back on extensive work with both computational environments, and not on a systematic study in which learners' interactions with equivalent tasks are compared. This study intends to throw further light on to this debate, examining what actually happens when students work on similar tasks in learning systems incorporating Logo or Cabri.

The entrance of the play paradox and talk of task as well tools indicate the need to consider more than the technical kernel of the microworld. Its design is just one of the aspects that need to be carefully developed. The idea that the microworld "does something" by its very existence has never been a viable one[6] and Balacheff and Kaput (1996) in their review of microworlds and mathematics learning conclude that,

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[5] She would probably now replace scaffolding with the notion of webbing (Noss and Hoyles, 1996; p.108-118)

[6] To emphasise this, Hoyles and Noss (1987) define a microworld in terms of three interrelated components: technical, pedagogic and cognitive. In most descriptions however, it is usually the technical kernel that is used as the defining feature (see, for example, Balacheff and Kaput, 1996). It is the microworld-in-practice that becomes a learning system constituted of the human participants, the technical kernels and the pedagogic intentions.

because of the play paradox, mere interactions with machine are insufficient to guarantee learning. They suggest that part of the answer to this paradox:

> "...lies in the search for design principles for teaching situations and teacher management involving microworlds where such characteristics could ensure the expected learning outcome."
>
> (p.483)

3.2.2 Instructional approaches and role of the teacher

The formal systems and phenomenological domains associated with a particular microworld may change the learners' mathematical experience at the epistemological level, but do not in themselves dictate the use of a particular instructional approach. In theory, it is equally possible to choose either the filling-outwards or the filling-inwards approach as described in the previous chapter.

Following the definitions given in *§2.4*, if the filling-outwards was adopted this would involve the development of microworld activities which make it possible for mathematically significant issues to *arise out of* user's own constructive efforts in the course of microworld interaction. In this approach, the teacher wants the student to take responsibility for producing knowledge as a personal response to a problem. The devolving of control of problem solution to the learner is behind the notion of the *adidactical situation* (Brousseau 1997) introduced in *§2.1.4*. The idea is that a problem is designed in such a way that the learner will, in coming to the solution, necessarily attend to the intended knowledge without it being made explicit by the teacher.

This does not mean that teacher interventions in terms of knowledge are completely nullified, but that they are reserved until the learner's personal knowledge has been acted with and expressed. The teacher can then intervene in ways aiming to illustrate how this knowledge can be fitted into, and perhaps re-expressed in terms of, the theoretical system of intended knowledge. In theory then, in the filling-outwards approach, the students construct their own knowledge mediated by the tools of the microworld and as doing so move outwards towards the intended knowledge.

The emphasis on mediation in the filling-outwards approach brings an aspect of the sociocultural school to its otherwise predominantly constructivist stance. A filling-inwards approach, however, leans more strongly towards the sociocultural, importing alongside the role of mediation, the idea that scientific concepts can be introduced to provide the basis of structures into which students can organise their developing knowledge. This implies a different action on the part of the teacher, whose aim is to make it possible for mathematically significant issues to become *appropriated during* learners' own constructive efforts in the course of microworld interaction.

Rather than starting by devolving the problem to the students, in the filling-inwards method the intended theoretical knowledge should be introduced before the interaction with the microworld. This would involve presentation of a teacher-generated mathematical voice, introducing knowledge situated in terms of a relevant theoretical reference system of the culture. The idea would not be the imposition of a particular learning trajectory in which this was imitated parrot-fashion until memorised, but that learners successively concretise the intended knowledge during microworld interaction.

According to Noss and Hoyles (1996; p.70), it is the idea of adidactical situations that converges with the view of pedagogy described in much of the literature concerned with microworld activity. This suggests that the dominant instructional approach until now has been the filling-outwards one. And, although there is a general agreement that the teacher has a crucial role in motivating, sustaining and monitoring microworld interaction (see for example, Balacheff & Kaput, 1996; Noss & Hoyles, 1996; Jones, 1998), perhaps it is because devolution to students has been such a dominant focus of research efforts that teachers' knowledge-related interventions have received less attention.

When Noss and Hoyles (1996) consider from a theoretical viewpoint the role of the teacher, they talk of a dialectical relationship between intervention designed to explain and elaborate and that designed to probe and understand. They are dismissive of intervention that involves telling or transmitting. Yet, the difference between explaining and telling might not always be miles apart and it is as yet far from clear as to which of the types of teacher invention are likely to be most productive in connection to microworld activities, leading Noss and Hoyles to conclude:

> "We require more precision as to what the teacher might say and do during the students interactions, and how they are contingent on students' actions."
>
> (p.70)

3.2.3 Four learning systems

These considerations of the research literature on mathematics learning with computer-based microworlds, especially taking into account the instructional approaches outlined in the previous chapter, mediated the decisions about the empirical work that will be undertaken in this study. Basically the study intends to address the issue of knowledge mediation, both in terms of teachers' structuring of learning systems and the structuring roles of different means of mediating human-computer interaction.

To this end, it involved the design of four learning systems incorporating microworld activity. The evolutions of mathematical meanings within each system will be traced by analysing what happens when groups of six students interact within them. Given the preceding discussion, the design will involve consideration of the knowledge students can be expected to bring to the system, the building of appropriate technical kernels and a detailed planning of the teaching interventions, both surrounding and during learners' microworld interactions. The evaluation of the system will need to look not only at how variations in the technical kernels and instructional approaches differentially shape learners' activities, but also how the learners act back on the system depending on these variables and on interactions between them.

In order to facilitate comparisons of a qualitative nature between the learning systems in action, a specific mathematical focus was chosen *a priori*, with the idea that the mathematical content be kept as consistent as possible across the four systems, although its meaning will be mediated by the tools. The mathematical focus chosen is that of transformation geometry, and more specifically, the isometry *reflection* (reflective or axial symmetry).

3.3 The mathematical focus

To provide a background into some of the general issues related to the learning of geometry that need to be taken into any account, before the specific mathematical focus of this study is addressed in detail, a brief consideration of theories of geometry learning is presented.

3.3.1 Geometry, the science of space

> "The branch of mathematics concerned with the properties of space and of figures in space."
>
> (Penguin Dictionary of Mathematics, 1997; p.185)

Geometry has both a practical and theoretical nature. In part, it attempts to model physical reality, but it is not predominately an empirical science (Freudenthal, 1973). As a mathematical practice, geometry, or rather geometries, are theoretical systems with specific discourses, their own axioms, rules of transformation, objects and problems (Laborde, 1993; Hoyles, 1996; Mariotti & Fishbein; 1997). This means that if we consider, say, a two dimensional figure in Euclidean plane geometry, then

> "The nature of this figure is double: they are material entities traced on a paper, earlier on sand, more recently on the screen of computers but they are also objects of a theory, resulting from an abstraction of reality. The widthless mathematical straight line belongs to the world of "idealities"."
>
> (Laborde, 1993; p 49)

Referring back to Plato, Laborde suggests that this double nature is illustrative of a dual role: the term *drawing* denotes the material entity, a specific configuration of elements expressed in visual form, while *figure* is reserved for the theoretical object, the geometrical elements and relationships which provide a text by which the visual form might be described. The nature of this text will depend on the geometrical system that is being used a reference. For the learner, then, an important aspect of understanding geometry is the co-ordination of the visual with the theoretical. There are a number of different theories about the ways in which may learners go about achieving this co-ordination.

3.3.2 Three theories of geometry learning

One of the best-known models concerning the learning of geometrical reasoning is the van Hiele model (van Hiele, 1959; 1986). According to this theory, learners pass, or rather can pass as a result of appropriately aimed teaching, through a hierarchical sequence of levels of thought in geometry from a "gestalt-like visual level through increasing levels of description, analysis, abstraction and proof." (Clements & Battista, 1992; p.426).

At the first level, the *visual* level, learners identify and operate on figures according to their appearance. At the *descriptive/analytic* second level, learners recognise and can characterise figures by their properties, but do not see relationships between classes of figures. Third, comes the *abstraction/relational* level when learners can distinguish between necessary and sufficient sets of conditions, can classify figures hierarchically and produce logical justifications. The fourth level is the level of *formal deduction.* Learners using this level of thought can establish theorems and produce proofs. In the final level, the level of *rigour* and mathematical thought, learners can reason formally about mathematical systems "in the absence of reference models" (Clements & Battista, 1992; p.428). Wirszup (1976) writes how "a person at this level develops a theory without making any concrete interpretation" (p.79). Hence, the overall picture in this theory is of visual considerations "falling into the background" (Clements & Battista 1992; p.427) as learners traverse upwards through the levels.

The van Hiele theory is useful in that it emphasises the varied demands of understanding geometrical systems as mathematical structures. It is reminiscent of the hierarchical theories presented in *§2.1.3* with what was intrinsic at one level appearing in an extrinsic manner at the subsequent level. Like these theories, there seems to be much emphasis on abstraction with little attention to concretion. Taken at its extreme, it seems even to have been interpreted in a way that denies any need for concretion. It is difficult to imagine how activity devoid of any reference model, as described at the level of rigour, could possibly have meaning. It is not that the reference model need correspond to some sort of experienced reality, or even to be external to the axiomatic systems, but some sort of reference model is surely needed.

The equating of development with the replacement of "lower" level models by "higher" level ones – with visual concerns giving way to axioms, definitions and theorems – carries the implication that interpretation of visual entities is unproblematic and that "gesalts" are not shaped by theoretical concerns and do themselves require development. This raises the question of precisely what is behind the visual prototypes used by students at level 1. Can we be certain learners are not evoking common geometrical properties, even if they do not have the means to explicate or express them?

Even if one was to agree that learning an appropriate means of expression is precisely what is indicative of a move to level 2, there are still problems. Would a learner with, say, a dynamic visual view of a rectangle, who saw a drawing of a representative of a class or rectangles, be considered at the same level as one who saw the same rectangle as a static drawing because neither explicitly referred to lists of properties?

Duval (1998) is critical of the van Hiele model on precisely these points. He argues:

> "There is no developmental hierarchy between different kinds of cognitive activities: visualisation, natural discursive reasoning, theoretical deductive reasoning, formal axiomatic proof, analytic or synthetic processes. In fact since the representational level (about the age of 2-3 years) until the most mature levels, we have visualisation, speech, reasoning, analytic and synthetic processes. But the way of working of these different models is not the same at each level and becomes more and more complex."
>
> (p.49)

To perform successfully in geometry contexts, Duval is arguing that learners must develop not only in terms of theoretical deductive reasoning, or even more everyday discursive reasoning, but also in terms of visualisation. Development of thinking is *multimodel* and not *unimodel.* He points towards what he calls a primitive duality of cognitive modes (Duval, 1999), with visualisation stemming from images and deductive reasoning with it original roots in language and concludes that their eventual synergy, and not substitution of one by the other, is necessary for proficiency in geometry. Nonetheless, before this co-ordination can occur

> "...work on differentiating between different visualisation processes and between different reasoning processes is needed in the curriculum, for there are various of ways of seeing a figure; in the same way there are various kinds of reasoning."
>
> (Duval, 1998; p.39)

Fischbein (1993) also thinks that geometry requires the development of cognitive entities that bring together mental processes of both verbal and visual nature. He too describes how geometrical concepts involve figural aspects relating to the modelling of space and theoretical "conceptual" aspects relating to their definition in a geometrical reference system. However, rather than advocating what he calls a "dual code theory" (p.152), he posits the existence of a third type of mental entity, the figural-concept.

> "To manipulate an image, a spatial representation under the strict but also intrinsic control of a definition would not be possible if only two independent processing codes exist. When solving a geometrical problem we manipulate the geometrical figures as if they were homogeneous mental entities."
>
> (Fischbein, 1993; p.153)

A figural-concept can be seen as the synthesis emerging from a visual/theoretical dialectic. For Fischbein, conflicts and difficulties experienced by learners of geometry can be related to the absence of harmony between these two aspects. This would suggest that at all levels of geometry learning, the aim should be the engendering of a harmonious synthesis between spatial and theoretical properties. Such a harmonisation, according to Fischbein, cannot be expected to occur spontaneously (Fischbein, 1993; p.156; Mariotti & Fischbein, 1997; p.220). On the contrary, it depends upon teachers' intervention. So, although it is important to consider progression in both visualisation and reasoning as Duval (1998) suggests, Fischbein concludes that for the teacher, focus should not be on viewing the twin processes in isolation, instead:

> "The main tasks of mathematics education in the domain of geometry is to create types of didactical situations which would systematically ask for a strict co-operation between the two aspects, up to their fusion in mental objects."
>
> (Fischbein, 1993; p.161)

Fischbein is clear that the teacher has an important role in mediating students' developing ideas about geometry. He also attends to representational systems, although perhaps more to internal representation than to external expression. In geometry, figures clearly represent an important external means of mediation, but as described earlier, when figures are interpreted as static drawings, they are easily detachable from their precise theoretical make-up. Some properties may be ignored, forgotten, unseen or unconsidered as students rely on perceptual cues alone to resolve problems (Hillel, Keiran & Gurthner, 1989). At the same time, certain particular instances of a figure can easily turn into paradigmatic models (Fischbein, 1987; Mariotti & Fischbein, 1997) or prototypes for the whole class (Hershkowitz, Ben-Chaim, Hoyles, Lappan, Mitchelmore & Vinner, 1990). In this case, learners who "know" the definition of a geometrical objects, but associate it with a specific drawing (or even set of drawings), have great difficulties in actually mobilising their mathematical knowledge (for a summary see, Clements & Battista, 1992).

The teachers' task is therefore most complex one and an important aspect of it involves making accessible to learners ways of expressing their knowledge that support the simultaneous consideration of visual and symbolic concerns.

One more point that can be made is that the particular geometry with which the theories in this section are concerned appears to be traditional Euclidean geometry (this is especially so with respect to the van Hiele levels). In terms of Piaget and Garcia's historical analysis of the development of geometrical ideas (*§2.1.3*), this geometry is based predominantly on intrafigural analyses, while transformation geometry, at least in the sense considered by Klein, is associated with the transfigural level. The study of geometrical transformations in their own right, following the epistemological classification of Piaget and Garcia, should involve considerations at both the intra and interfigural levels, since transformations involve looking beyond relationships within a figure to relationships between figures in a mathematised space.

Research relating to the learning of transformations is still relatively under-represented in the mathematics education literature, and as Malara and Iaderosa (1997) point out, even studies concerning the plane isometries, which have received more attention than other transformations, are relatively hard to find (p.208). Nonetheless, as will be addressed in the following section, in the studies that have been conducted a fair amount has already been learnt about the ways students negotiate (or fail to negotiate) the general problems involved in developing and expressing geometrical understandings related to transformations.

3.3.3 *Transformation geometry and reflection*

From a mathematical point of view, a transformation in plane geometry can be described as a mapping of the plane. The particular transformation reflection is one of the group of isometries involving one-to-one mappings of the plane onto itself in which distances between points are preserved (Coxford & Usiskin, 1971; Coxford, 1973; Willson, 1977; Thompson, 1985). The property that distinguishes reflection from the other isometries is that the mapping leaves an image point and its "pre-image" point equidistant from a line (Thompson, 1985; p.213). Figure 3.1 presents in visual and symbolic form a summary of the properties associated with the transformation (adapted from Coxford, 1973; p.139):

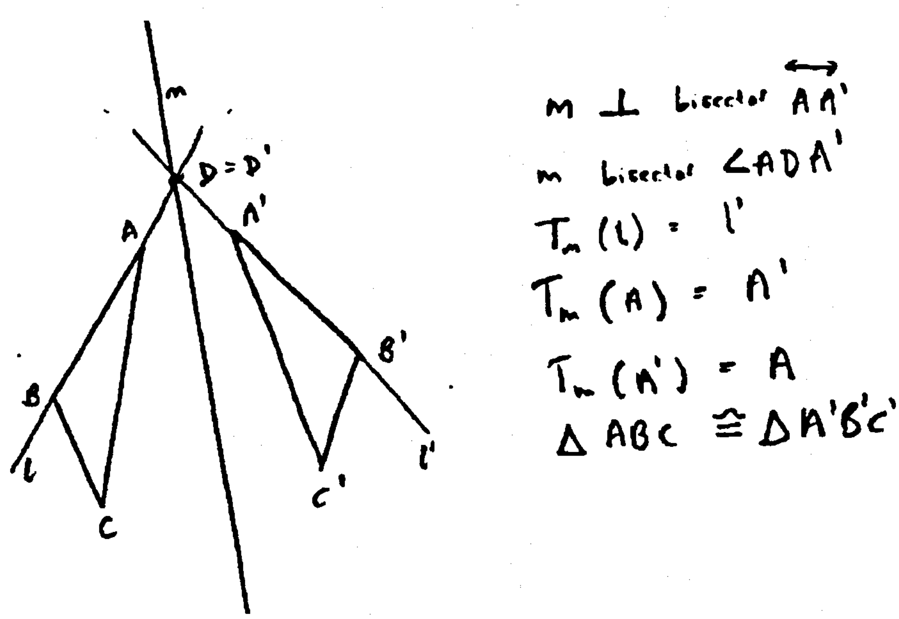

Figure 3.1:Properties associated with the reflection transformation

From an analysis of this figure, the following knowledge about the transformation can be highlighted:

- The image of a point (D) on the axis of reflection (or the line of symmetry or mirror line) is the point itself ($D = D'$). Points on the axis can be considered invariant under the transformation.

- The axis is the perpendicular bisector of any point and its image (m perpendicular bisector AA').

- The image of a straight line is another straight line ($T_m(l) = l'$), and the axis is the bisector of the angle made at the point at which a line and its image meet (m bisector $\angle ADA'$).

- A geometrical object and its image are congruent $\Delta ABC \cong \Delta A'B'C'$.

- The transformation reflection is a self-inverse transformation ($T_m(A) = A'$, $T_m(A') = A$).

- The transformation reverses the orientation of non-collinear points (in ΔABC points A B C (in that order) are anti-clockwise, in $\Delta A'B'C'$ the respective image points are clockwise).

Above the properties are presented as a list, but not all are required to define the transformation. For example, Definition 1 below rests on the property of the axis as perpendicular bisector, while that in Definition 2 uses the axis as bisector.

Definition 1: Given a line m, a reflection with m as axis can be defined as the transformation T_m: $\Pi \rightarrow \Pi$ such that $T_m(P) = P'$ if and only if $PP' \perp m$, $m \cap PP' = \{A\}$ and $PA \cong P'A$ $\forall P \in \Pi$

The first definition was adapted from that provided in Jaime and Gutiérrez (1996; p.27). In terms of the geometrical relationships explicitly included, this is the form most commonly found in mathematical texts and can be regarded as the classical definition of the transformation. It is by no means the unique definition, as can be seen in Definition 2 which can be regarded as equivalent in terms of independent and dependent variables despite relying on different geometrical relationships.

> Definition 2: Given a line m, a reflection with m as axis can be defined as the transformation $R_m: \prod \rightarrow \prod$ such that $R_m(P) = P'$ if and only if m is the bisector of $P\hat{A}P'$ ($\forall A \in m$ and $A \neq P$) and $PA = P'A$ $\forall P \in \prod$

To put this another way, if the isometry transformations are considered as analogous with their algebraic counterparts, functions, then two reflection functions can be considered as equivalent in they produce the same net result in terms of the initial and final state of all elements of the plane, i.e $T_m(P) = R_m(P)$ (Leron & Zazkis, 1992).

Having presented reflection from a mathematical point of view, it remains to be seen how close or how far this knowledge corresponds with that mobilised by students when they work on mathematical tasks related to this transformation. One way to examine this question is to consider students' paper-and-pencil constructions.

3.3.3.1 Drawing reflections

Figures 3.2 and 3.3, present four more diagrams associated with reflection.

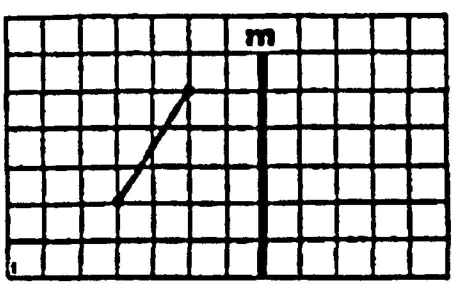

Figure 3.2a: Task involving construction of an image under reflection transformation

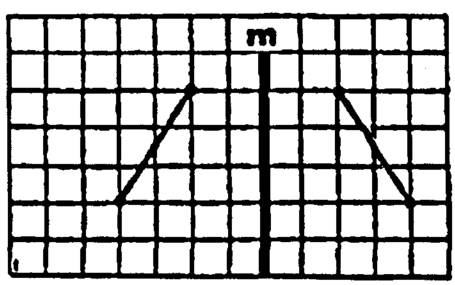

Figure 3.2b: Sketch of the image

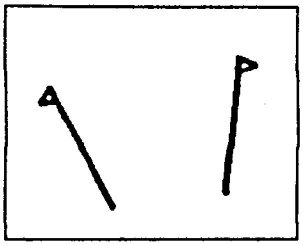

Figure 3.3a: Task requiring construction of missing axis of reflection

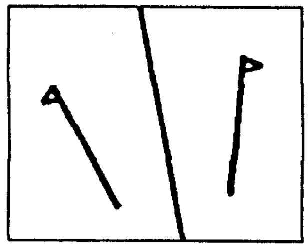

Figure 3.3b: Sketch of the axis

The first observation that can be made about both examples is that the sketched image can be considered to be the intended response. The second is that most secondary age students can be expected to produce the intended responses when presented with items similar to them (see, for example, Küchemann, 1981; Grenier, 1987; 1988; 1990). The two items have actually been borrowed from the Küchemann (1981) study, in which 1026 students completed a test including 27 items concerned with reflections. He found that more than 80% of 13-15 year-olds answered the first item correctly and over 70% drew an appropriate axis.

Looking at the first item as it was presented to students (Figure 3.2a) in more detail, it is obvious that it is rather particular. The plane has been represented with a grid of horizontal and vertical lines. The axis for the reflection is vertical and the segment has its two vertices on intersection points of the grid. From a strictly mathematical view, the segment ought to be viewed as a particular subset of the plane which has be "highlighted" to show the effect of the transformation of the whole plane (Thompson, 1987), the grid should be seen as underlying the plane, and not of another highlighting of point sets. In terms of the levels introduced in *§2.1.3*, this task can be classified as involving interfigural concerns, because the intended focus is on relationships between objects (although intrafigural concerns might enter into any validation activities).

The second item (Figure 3.3a) could be considered as either of the intrafigural type, since it can be solved through actions within the objects, or the interfigural one, if attention is rather on the transformation that maps one on the other. In its presentation, the task appears of a more general nature then the one in Figure 3.2a, at least in that none of its elements have been drawn at the horizontal or vertical and the item has been presented on plain rather than squared paper.

Despite different presentations and different demands, most students could do these problems, suggesting that the majority of learners of secondary school age do know something about reflection. But when the evidence from other rather differently presented items is also taken into consideration, it does not seem that students' sketches were produced on the basis of mobilising robust knowledge of well-defined transformations of all the elements of the plane onto itself. Even if the plane is left aside and the emphasis is on the mapping of one geometric figure onto another[7], it seems that rather than mathematical definitions of invariant properties, student responses during the studies listed above were more influenced by specific features of task presentation.

For example, Grenier (1987) considered students' performance on 96 items presenting an axis and a segment grouped according to systematically varied task features (orientation of axis, angle between segment and axis, intersection of axis and segment, use of squared or plain paper). She found that errors become common when the axis of reflection was neither horizontal or vertical and increased further as the angle between this axis and the segment was made larger. Where the segment crossed the axis, errors were also more frequent than when the entire segment appeared on one side of it.

By analysing students' incorrect responses, both Grenier and Küchemann found that many errors were systematic and result from the application of procedures with one or more incorrect steps. A common mistake, especially when aspects of the original figure are horizontal or vertical was to seemingly ignore the actual orientation of the axis and reflect across or down the page. Despite being influenced by visual factors, the systematic nature of student procedures makes it difficult to classify them in terms of the van Hiele levels presented earlier: students did operate on figures according to

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[7] According to Coxford (1973), this may be considered appropriate until group characteristics of certain sets of transformations are to be discussed (p. 161), i.e., when students work on activities of a transfigural nature.

their appearance (level 1), but their attempts to mobilise properties (level 2) led to drawings that don't necessarily look like reflections.

Küchemann discusses a strategy which he calls *semi-analytic* in which one point of the image was correctly located but then the orientation of the axis forgotten when the rest of the image was sketched. Grenier mentions another procedure, involving translation of the original segment or its extension through the axis. Bell (1993) highlights what he sees as another "misconception", pointing to a tendency among school students to accept more than one possible image as correct for the same figure in the same axis (p.131-132), apparently violating the property of reflection as a one-to-one mapping.

Although the focus in these studies is on errors, actually all these methods mobilise some appropriate geometric properties. In particular, students seem to know to preserve (some) distance and, usually, where the pre-image is more complex than a single segment, they attempt to represent the opposite or reversal property. Moreover, both researchers also indicate that some students do use orthogonal procedures, although Küchemann reports that very few students, even when they were asked, use terms such as "perpendicular", "90°" or "at right angles" (p.141) to describe the relationship between a segment joining a point to its image and the axis.

Success on items like that presented in Figure 3.3 is also seen as no guarantee of a conscious awareness that the axis is invariantly the perpendicular bisector of a point and its image under reflection. According to Grenier (1988; 1990), many student construct the axis by considering midpoints of two sets of points and their images and then draw lines through these point, constructing equal distances but not angles. This suggests the task was treated at the intrafigural level, with students analysing relations within the figure as a whole, rather than using an external reference to mathematically structure their relative positions. Evidence from the Küchemann study supports this view, as he found that rather than recognising items in which no axis of reflection could be drawn between two congruent objects, many students adopted the same midpoint strategy (p.147).

The overall picture presented from these studies is one of inconsistency in strategy and sensitivity to task presentation. The students appear to have interpreted the tasks as about the production of a series of specific visual images rather than the development of a general approach.

Thompson (1987) writes of how he feels a paper and pencil approach is more or less doomed to failure. Describing his attempts to have students (in this study teacher-trainees) develop straight edge and compass constructions and apply these procedures to problems of mapping points in the plane under isometric transformations, he writes

> "That approach was singularly unsuccessful, for two reasons. First, students would become absorbed in a construction's details and lose sight of the mathematics they were supposed to learn. Second, they considered each application of a procedure as a unique transformation, even if it was applied as a subprocedure."
>
> (Thompson, 1987; p.94)

This meant, for example, when asked to find the image of a triangle under a particular transformation, they felt three transformations rather than one had been applied (one for each point of the triangle). Reflection, in this view, is not of a single mapping (either on plane to plane or figure to image), but a set of x transformations, where x is the number of vertices of the pre-image – or even of nx, where n is the number of particular properties constructed for each vertex.

There is a sense when reading of students' strategies that they are only to be expected. On the one hand, without knowing the invariant properties of a transformation, how can student validate the visual images they are being asked to produce? On the other hand, how well can students illustrate that they know the invariant properties, if they do not represent them visually? Are they hence left in a position from which they can neither abstract nor concretise?

One possible response to this paradox is to provide students with the means to take care of the constructions underlying the mapping. This could enable them to obtain visual images that result from applying the reflection transformation, before they know about its mathematical properties, the idea being that these could then be abstracted from the visual products. There have been a number of studies focussing on the activities of students when they are given opportunities to investigate visual representations of transformations. They can be divided into three main groups: transformation as physical actions, tools for transformations and microworld explorations of transformations.

3.3.3.2 Transformation as physical actions

Various researchers have pointed to the link between the isometry transformations and physical actions (see, for example, Willson, 1977; Coxford & Usiskin, 1971; Küchemann, 1981; Edwards & Zazkis, 1993; Lehrer *et al.*, 1998). Yet, there is little evidence that such work helps students in constructing robust figural-concepts in the transformation geometry domain.

For example, in a recent study, Lehrer *et al.* (1998) investigated the development ideas of about transformation geometry amongst young students (aged 8 years) in the context of designing quilts, a culturally familiar object which provides everyday instantiations of symmetrical patterns. As panels for their quilts, the students used patterned squares (with an identical pattern front and back), which were flipped and rotated to produce four square designs or strip patterns. They were encouraged to develop notations to describe their actions systematically and hence investigate compositions of transformation. Lehrer *et al.* report that students experienced difficulties in linking notation and action, tending not to notate comprehensively their physical actions. Such activities appear to have led these young students to realise the limits of empirical activity for justifying conjectures, though not to engage in more analytic argument. This is not altogether surprising. The properties of the transformation are not made explicit through flipping and turning actions; hence students do not have easy access to the geometrical arguments that could be used in analytic justifications.

Obviously too these actions are some distance from thinking about operating on sets of points and further still from ideas about mappings of planes onto themselves. The intended knowledge associated with these activities is different, and, in particular, the invariant geometrical properties of the general reflection and rotation transformations are not the focus for learning. Instead, attention is on specific "local" reflection and rotation transformations (reflection as "flip" around one side of a material object, rotation as "quarter turn" around corner of object). Edwards and Zazkis (1993) describe such specific transformations as corresponding to "primitive" or "naïve" conceptualisations of reflection and rotation; that is, as ideas held before prior any instruction in transformation geometry. This would suggest that Lehrer *et al.* selected an appropriate starting point, they call them "springboards" for students learning, but there is also a danger. If these instantiations of the transformation are over-emphasised in the absence of other examples, students will not be challenged to modify their knowledge or connect it with alternative conceptualisation more closely related to the intended "official" mathematical knowledge.

3.3.3.3 Tools for visualising isometric transformations

Physical action even on "two-dimensional" material objects can have the effect of removing them from the plane. The same is true of activities in which students use folding as a means to draw images under reflection. Grenier (1987) suggests that the act of folding can lead students to think "reflection is a geometric transformation from one half-plane to the other" (p.187). A similar problem can occur if mirrors are placed along the axis for reflection and used as tools to produce the visual image (Zuccheri, 1998). Mirrors can also be associated with another difficulty. To be used successfully, a mirror must be positioned along the axis and perpendicular to the plane. If it is tilted, the position of the image changes. It could be that this is one reason that students have been observed to accept more than one possible location for an image of a given object in a given axis (Bell, 1993). It was in response to problems such as these that Giorgolo created a didactic tool called the *Simmetroscopio* (Zuccheri, 1998).

This tool has been described as a collection of semi-transparent mirrors, which can be assembled in different ways to produce images of various transformations. Its advantage over an ordinary mirror is that it overcomes the half-plane problem: images and pre-images can be seen simultaneously on either side of the axis. Yet, in

Zuccheri's description of didactic activities, it seems that the geometrical properties the tool highlighted were those to do with distance and opposite congruence – the very properties that students are most able to express in their own paper and pencil constructions. Orientation of images in relation to the axis and preimage is still not a salient concern. Despite of its advantages over ordinary mirrors, this physical tool still takes care of the mathematical constructions in a way that does not make available explicit access to their properties. This may help student visualise but not theorise about the transformation. In Fischbein's terms, figural aspects dominate and a harmonious fusion with theoretical aspects is not motivated.

More evidence for this can be found in the study by Malara (1995). She investigated whether the opportunity to visualise the effects of different isometric transformations promoted the formation of robust figural-concepts. In her study, a computer produced the visual images. Following "moments of visualisation", paper-and-pencil tasks were given to the students. Malara reports that despite some improvement, and in particular the overcoming of the classical treatment of horizontal and vertical segments, students still had considerable difficulties in operationalising and making explicit all the relevant properties of the isometries with which they worked (translations, rotations and reflections).

Maybe it is making explicit that holds the key to interpreting students' difficulties, which brings back the notion of formalisation (see *§3.1*). Using computer technology, it is becoming technically easier and easier to create stunning patterns and designs by applying isometry transformations (see, for example, Graf & Hodgson, 1998). But if there is little emphasis on formalising the geometrical properties that constitute these patterns, then rather than engaging with mathematics, the tools could be used by students in ways that more or less avoid it. Using sophisticated computer tools, like using mirrors and like folding activities, perceptual concerns can be considered entirely separately from geometric ones. It may be true that the tools provide students with the data from which to abstract general relations. The difficulty is that, since the goal is associated with the visual, this abstraction does not appear to happen in any systematic way.

No way out of the paradox has yet been found. The tools described in this section permit students to construct images under transformation without expressing their geometric properties. And students are expected to express the products of the transformation visually. They might be mistaken into believing that transformation tasks are about memorising and reproducing visual configurations, not constructing general relationships.

Now, recalling that microworlds offer a tool-set of modifiable formal objects and relationships, along with graphical representations of these elements in action, it might be hypothesised that it is just at this point, the connecting of the visual image and its formal description, that they might offer an appropriate way to resolve the paradox.

3.3.3.4 Microworld explorations of transformations

A number of researchers have addressed students' learning of transformation geometry as they work on computer-enhanced activities using specially customised computer-based worlds. These studies have included worlds based upon both Logo and Cabri (the softwares considered in *§3.2.1*). Strictly speaking, these computer activities were not always structured in ways that correspond to all the aspects of microworlds as outlined in *§3.2*. Nonetheless what all have in common, is that they represent computational worlds in which students can interact with visual representations and formal systems.

In terms of the technical kernels of the system, there is, not surprisingly, considerably more variation amongst the five Logo worlds chosen for consideration, than in the Cabri ones. Because Logo is a programming language, the designers had a less constrained choice of the particular geometric model that would be embodied in the technical kernel, with the possibility of building activities that would emphasise ideas associated with turtle geometry, Euclidean geometry and/or co-ordinate geometry (see Kynigos, 1992).

Logo models have been based on the classic intrinsic turtle geometry (Gallou-Dumiel, 1987; Leron & Zazkis, 1992). Or, following Loenthe (1992), on a multiple-turtle geometry which included some extrinsic elements (Hoyles & Healy, 1997). A model emphasising aspects of co-ordinate geometry was built by Edwards (1991; 1992; 1993) and Thompson's (1985; 1987) microworld incorporated aspects of co-ordinate geometry along with aspects of turtle geometry. The microworlds build by Gallou-Dumiel and Healy and Hoyles were limited to the exploration of the one isometry, reflection, whilst the others considered a group of isometry transformations.

One of the most striking things when considering all of these microworlds is that none of the activities involved students in actually building from scratch a general procedure for any of the isometric transformations.

Edwards and Thompson both provided ready-made tools to execute isometric transformations as primitives in their microworlds. They are also similar, in that the placing of the screen objects to which the transformation will be applied was controlled by the system not the user. This may be an important feature given the sensitivity of learners to particular task aspects, such as the orientation of the figures in the plane and of the axis for reflection.

Edwards's TGEO (Transformation GEOmetry) emphasises transformation from the perspective of co-ordinate geometry. For example, the REFLECT tool is executed by typing three inputs to define the location of the axis, the x and y screen co-ordinates and global heading (measured from north as 0°). The visual effect of transforming the plane is highlighted by the presence of an L-shape whose "before" and "after" locations can be compared.

Edwards was hoping that students using her microworld would come to view transformations as mappings of the plane onto itself and, although in the analyses of users' interactions with the TGEO, little explicit attention was directed towards examining the impact of instructional approach, she seems to have experimented with the use of both filling-outwards and filling-inwards methods to support the development of this view.

The student teachers who used the microworld in a filling-outwards manner began with a set of activities involving actions on physical objects (Edwards & Zazkis, 1993). In contrast, in other studies (Edwards, 1991; 1992), TGEO interaction was preceded by an introduction to theoretical issues, in which a concrete model of transformations of the plane was provided. Edwards (1992) described this as a "dual-plane" model. She wanted students (in this case aged 11-14 years) to conceptualise an infinite moving plane to which any transformation is applied, and a stationary reference plane behind it (p.147). Her concrete model of this view involved two sheets of paper, the moving plane initially superimposed on the stationary plane and then rotated, folded or translated according to which transformation was applied. Edwards's contention was that the need to be explicit about the effects of particular inputs in the computational versions of the transformation would aid students in connecting the TGEO feedback with this dual-plane model.

In practice, she reports, many of the students used a rather different mental model, in which a single turtle exists in a single plane and moves about within it via commands given relative to its current position and heading. Her intention was that the visual feedback in TGEO would help students move from this "misconceived" notion to one involving visualisations of movements of the whole plane. The results she reports however suggest that, rather than concretising the dual-plane notion that was introduced prior to computer interaction, students' were more likely to further concretise their original view.

Like TGEO, Thompson's MOTIONS microworld also made extensive use of the co-ordinate system underlying the Logo screen and included transformation tools amongst its primitives. It differed from the TGEO world, however, in that its primitive rotation and reflection tools were specific rather than general versions. The tool FLIP has the effect of reflecting the plane through the y-axis, for example. MOTIONS can be considered more truly a microworld than TGEO, since users are expected to extend the primitive tool-set with procedures for compositions of transformations and finally general reflection and rotation tools.

Thompson developed a curriculum to accompany his microworld in which activities were divided into three groups. The first group involved considering effects of a given transformation, the second involved generalisations of transformation effects and the third transformations as objects. As such his activities can be described as moving from intra-considerations, through inter, to transfigural issues, and as following a filling-outwards instructional approach. He claims that the proposed curriculum proved "very difficult" for student teachers (1985; p.230). Like Edwards's subjects, they too tended to imagine the transformation commands as operating on a figure and not the plane, and this particularly made the final activity-set hard to complete.

So it would seem that providing users with these ready-made tools for transformation and encouraging them to formalise relationships by manipulating them did not result in an easy appropriation by the students of the mathematical ideas intended by the designers. One possible reason for this is that, when interacting in these two worlds, the students had no way of knowing that the effects of the transformation were even more general than the visual feedback indicated. At the screen level, they were presented with figures and with tools that operate on these figures. There was not much reason to think of the figures provided as sets of points, let alone to conceptualise the plane in this way. Furthermore, although the activities focused on formalising aspects of transformations, they did not explicitly relate to formalising the construction process underlying the given transformations. In this respect, the tools were opaque to the users.

While users of TGEO and MOTIONS worlds worked with a subset of specially created Logo commands intended to stress global aspects of polar and Cartesian co-ordinate geometry as much as (or more than) the instrinsic geometry of the turtle, Gallou-Dumiel's work involved interaction with the basic primitives of turtle geometry. She was interested in how the presence of computational instruments provoked changes in students' activity. She asked students to work through a series of construction tasks in which they are given figures such as those in Figure 3.1 and asked to produce on the screen their images under reflection. In the Logo setting, she argues (Gallou-Dumiel, 1987), the most economical solutions involve the explicit utilisation of angles to operationalise the change in orientation associated with the reflection transformation. This is not necessarily the case in corresponding paper-and-pencil tasks (see also Laborde, 1990; p.334-335). Gallou Dumiel's activities at no point involved the construction of a single general Logo procedure. Instead, students were expected to construct specific commands for each given figure. She found that, despite the change in media, aspects of the task and its presentation continued to play a strong influence on the solution strategies adopted by the students, with many of the difficulties observed in paper-and-pencil setting emerging once again.

A problem in constructing reflections in turtle geometry microworlds is that students must have some way to determine the relationship between different elements of the plane. Gallou-Dumiel overcame this by specifying on the diagrams given to subjects the perpendicular distance of a point on the figure to the axis of reflection. The transformation tools in TGEO and MOTIONS presumably make use of the co-ordinate system underlying the Turtle-geometry screen. For our Turtle Mirrors world (Hoyles & Healy, 1997), we decided that students themselves should have some ways of ascertaining all the geometrical relationships necessary to construct an image under reflection. Like Gallou-Dumiel, our emphasis was on the properties of the reflection transformation, rather than transformations as mappings of the whole plane, and we too presented students with a sequence consisting of sets of specific items, which began with the construction of images when Logo code was available as a support and went on to the construction of methods to produce images in the absence of such support. The sequence did not get as far as activities involving the construction of a general Logo procedure.

In Hoyles and Healy (1997), we described the variety of different but correct strategies that students used to construct a particular image under reflection. In contrast to previous studies in all of which correct methods seemed to be associated with the use of perpendicular constructions, we found in the Turtle Mirrors setting students tended to make use of other relationships. Where students had access to the Logo code behind a screen design, the process of constructing an image of the design under reflection became the almost trivial one of swopping left and right turns. In this case, we found student can easily co-ordinate the visual and symbolic aspects of their constructions[8].

The Logo language had an important mediating role and in its the absence, students attempted to find ways of reconstructing it by measuring and constructing equal angles and distances between the drawing and the axis. Although we found students could come up with novel and mathematically consistent methods to produce images under reflection, like the students in Gallou-Dumiel's study, when the code was taken away, the majority of our students also modified their strategies according to task features.

Both these studies suggest that, if there is no explicit emphasis on constructing a general procedure for a transformation, students will change methods according to task presentation. Another point they have in common is the adoption of filling-outwards instructional approaches: in neither are students introduced to the intended theoretical concepts prior to computer interaction.

Leron and Zazkis (1992) do emphasise a general method for reflection. Their focus is on group theory and they are concerned with the definition of a turtle group to represent turtle motions in a two-dimensional plane. A third Logo model for a plane is defined by these two researchers. They suggest that, if Logo commands are viewed as operating on turtle states (consisting of their position and heading), then a turtle plane can be described as the set of all turtle states. This is more attractive than Edwards' dual-plane metaphor, since it seems much closer to the way users usually relate to turtles. In practice, it means that, if users define an operation with reference to one turtle, they have to bear in mind that the same relationships hold for the turtle in any other state. In this manner, a screen turtle can be treated as a generic example, just as specific numeric examples might be in the case of algebra (see, Mason & Pimm, 1988; Balacheff, 1988).

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8 Noss (1997) describes similar activities in another turtle geometry microworld which result in the same interplay between students' appreciation of symmetry at visual and symbolic levels (p.298).

Leron and Zazkis themselves came up with formal definitions of the isometry transformations in terms of operations on turtle states. They describe how using these definitions, various notions from group theory can be addressed. A reflection is applied to a turtle state in the following way:

i. a translation through a given distance <RT *a* FD *b* RT -*a*>

ii. a right rotation through an given angle <RT *k*>

iii. a *flip* in which a turtle leaves the plane, rotates through 180 degrees and returns belly-up to the same place (this has the effect of sending the turtle to the right instead of left and vice versa in subsequent commands) <FLIP>

iv. a right rotation through the negative value of the given angle in (ii) <RT -*k*>

v. a translation through the negative value of the given distance in (i) <RT *a* FD -*b* RT *a*>.

The procedure transforms an individual turtle from its initial state to its image state under a reflection in an axis defined relative to its initial location. A learner who wants to explore the effect of a reflection in the same axis of two or more turtle states – to test, for example, that the distance was preserved in the corresponding images – might however experience some problems if they adopted this procedure. Because distance and orientation of turtle state in relation to the axis is defined by angle a, distance b and angle k, for a second turtle state to be transformed under a reflection in the same axis, it would be necessary for the user to calculate the appropriate functions of a, b and k to produce the new values to be used for the second turtle state (see Figure 3.4).

This would presuppose that they already understood how the transformation was defined, rather circumventing the need for the microworld interaction in the first place. Once again a harmonious relationship between formal description and visual image is hard to obtain, although this time it is the formalisation that dominates.

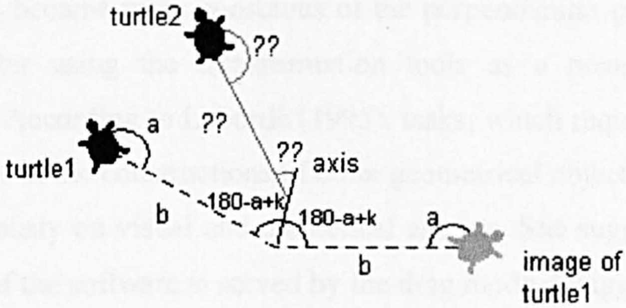

Figure 3.4: Difficulty of determining the commands for the transformation of the second turtle

So what can be learnt from the Logo work related to the transformation of reflection? First, the problem of designing a microworld in which mathematical knowledge of this mapping can effectively evolve is far from solved. Second, whatever the views users do construct as a result of their interactions with the computational tool, it seems that these views will be connected to the precise nature of the tools and the intended mathematical foci behind the tasks they work upon.

From reviewing the TGEO and MOTIONS worlds, it appears that, when tools that provide visual images are given, the associated tasks tend not to stress the formalisation of the invariant geometric properties underlying the tools. In the microworlds of Gallou-Dumiel and Hoyles and Healy, on the other hand, the problem of knowing that a correct visual image has been constructed by users who are not fluent with the definition of the transformation is associated with similar difficulties as with the paper-and-pencil items. It could be argued that the tools of all the Logo microworlds have been designed with an eye to construction but not addressed the problem of validating constructions. Is the same the case with work in Cabri microworlds related to transformation geometry?

In Cabri-géomètre, tools for transformations are given the same status as other primitive construction tools like perpendicular lines, parallel lines, midpoints, etc. governed by geometrical properties. Jahn (2000) suggests that this might be expected to lead to their systematic use in construction tasks (p.94).

Some support for this conjecture can be found in the work of Guillerault (1991) who found that students became more conscious of the perpendicular property associated with reflection after using the transformation tools as a means to construct a perpendicular line. According to Laborde (1995), tasks, which require students to use transformation tools in the constructions of other geometrical objects, encourage them to focus simultaneously on visual and theoretical aspects. She suggests an important mediating feature of the software is served by the drag mode. Dragging points of their constructions disqualifies purely visual strategies, by illustrating how constructions can be "messed-up" (Healy, Hoyles, Noss & Hölzl, 1994) if an appropriate sequence of constructions is not used. Furthermore, when a visual-spatial relationship is not messed-up, students can attribute its behaviour to the tools they used in its construction, making the geometrical properties more salient.

In the tasks described by Guillerault and Laborde, students are directed towards the use of the reflection transformation in their constructions (see also, Capponi, 1993). In our work, we found that students sometimes discover the tool for reflection by themselves even in tasks for which we had not anticipated its use (see for example, Noss, Hoyles, Healy, and Hölzl, 1994). Nonetheless, students using a transformation tool as a means to an end, tend to stop their investigations at the point at which this end has been achieved. For instance, this might be at the moment that dragging activities indicate one line remains parallel to another as it is moved around the screen. So, while they might focus on properties of the transformation in the construction process, there is no need for them to formalise these properties for themselves.

We have also worked with tasks that more specifically address the properties of reflection, involving, for example, the construction of axis of reflection (described in Hölzl 1996; p.183 and Noss & Hoyles, 1996, p.115-116) or an image of a given figure in a given line[9]. This last task involved students in constructing, or at least in attempting to construct, a general method that could be used to reflect any point using

[9] Of course, because this is dynamic geometry, the "given" elements are variable by dragging.

any line. This was precisely the task that was lacking from the Logo activities. A successful attempt is described in Noss, Hoyles, Healy, and Hölzl (1994). It involves the use of a strategy common in Cabri, that of solving first an intermediate problem by "relaxing" one (or more) constraint (Goldenburg, 2001[10]). Specifically, the two students began by constructing equal distances, from pre-image point to axis and axis to image point using the circle tool, then fixing the perpendicular property by eye. This was done by dragging the mirror to a vertical position and creating a horizontal line through it. So far, the solution operationalises all those ideas that research has shown students to know about reflection. The mediation role of the software came to the fore at the point in which the student-pair (following a teacher intervention[11]) moved one of the lines. These two students knew that the two lines could be dragged independently of each other, and hence, that the image would be messed up. What is more, their dragging activities appeared to make the necessary formalisation, in this case the perpendicular property, more salient and the pair were able to complete a robust construction process which could be applied to other points on the screen.

In spite of this success, a question still remains as to what view of the transformation they were building. This issue was not specifically addressed in our study. Jahn (2000) cautions that even the *semi-pointwise* (or *analytic* as Küchemann describes it) approach used by these students can still lead to a notion of transformation that can be reduced to the idea that a figure can take up two different positions (Jahn 2000; p.95) and not emphasise functional aspects. She would like students to think in terms of an image under transformation as a figure F' formed from the complete set of points (not just the vertices) that are images of the set of points representing figure F (p.99). Her strategy is to leave aside the isometries and work instead with an affine

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[10] See also Love (1996), who describes this as "letting go" and Hölzl (1996) who describes this and other "dynamic solution" strategies. Relaxing could be considered as an example of a form of computer supported webbing which could be added to those described by Noss and Hoyles (1996; p.110-119).

[11] In all of the studies all of the studies involving exploration of transformation geometry in Cabri settings, the instructional approach used can be characterised as filling-outwards, although none of the studies have focussed specifically on this aspect.

transformation, with the idea that it is only when students work with transformations in which the form of image is not the same form as the original figure and especially when the image-form is not obvious by application of the transformation to its vertices alone, that they will be motivated to reconceptualise their views of both figures and transformations.

She seems to want students to leave behind the notion of operating on whole figures and replace it with the pointwise view. Thus she stresses the danger of working exclusively with transformations in which validation on the basis on congruency represents an important characteristic of students' approaches. A problem with this is that it seems to rule out the use by learners of the very property they seem to know most about. An alternative to leaving aside the isometries would be to could focus on connection as well as differentiation, with the idea that there are multiple ways of conceptualising geometrical objects. A whole figure view might provide an accessible means of validating a figure constructed using a pointwise approach. This means that an isometry like reflection could provide an appropriate mathematical focus.

3.4 Summary and Research Questions

This chapter has been concerned with the mediation of mathematical meanings by material and psychological tools. It began by introducing a particular kind of psychological tool, a formal symbol-system and associating mathematics learning with expressing formalisations of activity using the language of such systems. Special arenas for formalising activity, computational embodiments of mathematical notions called microworlds were then considered in *§3.2.* One way to examine the mediational role of tools is to observe if and how the evolution of ideas is differentially shaped according to the particular tool set that is available in a learning system. To this end, the rather different *technical kernels* of Logo microworlds based on turtle-geometry and Cabri microworlds in which Euclidian geometry is embodied were introduced. Turning to the other aspect of mediation, that of instructional approach adopted by the teacher, the way the filling inwards and filling outwards approaches *(§3.4.3)* can be used to structure students' interactions with microworlds was also considered.

The rest of the chapter concerned issues about the learning of geometrical notions. By combining aspects from theories of geometry learning, the development of geometrical thinking was characterised as an increasing fluency with, and co-ordination in figural-concepts of, visual and symbol expressions properties of figures and of the space in which they are contained.

More specific aspects of this development were considered in relation to the particular mathematics focus of reflection. In this respect, research indicates a fundamental difficulty for learners: knowledge of invariant properties of a transformation and their visual recognition are prerequisites for each other. This makes the design of teaching activities problematic. It is unreasonable to ask students to construct images under reflection if they are unable to recognise when they have successfully done so. On the other hand, activities which allow them to produce visual representation without expressing "texts" which describe the underlying properties, can have the problem of hiding unfamiliar ones and leaving them inaccessible to explication.

Despite this apparent paradox, the majority of learners do know quite a lot about the effects of applying the particular transformation of reflection to objects of a two-dimensional plane. Nearly all of them know that the same shapes should be represented in image and pre-image, most also seem to know to construct some distance between the axis of reflection and both image and pre-image as equal, and that figures appear in a reversed configuration. This knowledge does not, however, correspond to its precisely definable mathematical counterparts, and is operationalised differently depending on particular features of the task. In tasks that learners can do, there tends to be a complementary symmetry between the way geometrical properties are expressed and the visual outputs that derive from this expression.

One reading of the research literature presented above is that it indicates learners have more success when presented with tasks that can be interpreted from an intrafigural perspective, where additional constructions can be made on the basis of relationships internal to what is viewed as a single figure, than when interfigural demands involving co-ordinating relations of figures to aspects external to them are required. Transfigural interpretations, in which the focus is on reflection as an element of a structure of transformations are neither evident nor necessary in response to the kind of tasks commonly associated with reflection in secondary school mathematics classrooms.

Another classification useful in distinguishing between ways of thinking about a transformation is by considering views of the objects on which the transformation operates. First, distinctions can be drawn between three views of transformation (see Grenier & Laborde, 1988; p.66): as mappings of planes onto planes; as mappings of figure-objects onto figure-images; or as a relationship between two geometrical configurations or two parts of the same configuration. In the latter view, which could be aligned with an intrafigural analysis, the functional character of a transformation is absent. Here again, learners' views depend at least in part on the mediating resources of the learning setting; mirrors and folding activities emphasise acting on planes, manipulation of two-dimensional objects emphasis operating with figures.

Second, various possible meanings for planes and figures can be identified. Four alternative models of planes have been described: Planes as physical spaces in which elements exist; planes as flat surfaces containing a set of moveable elements; planes consisting of an infinite collection of homogeneous elements with a precise location in a two-dimensional system, and a dual plane model, in which there is a fixed two-dimension reference and a superimposed moveable equivalent. Similarly, four different meanings for figures have been proposed: Figures as whole objects that can be "picked up" in their entirety; figures defined in terms of particular points and the lines between them; figures as traces or trajectories on a display scheme; or figures as infinite sets of points.

From the research literature, it appears that some meanings are more common than others, and much of learners' activity suggests they are far more likely to see reflection as a relationship within a geometrical configuration than as a mapping from initial to final state and that they hold views in which planes are seen as physical spaces and figures as whole objects. Once again, however, the extent to which other views are accessible or not seems to be associated with the task demands and the means of mediation. For example, learners do not seem to spontaneously identify invariant properties of a transformation just because they have the means to produce their visual representation. And even when this is necessary for task completion, they do not seem to develop a consistent general approach. Instead, they construct an eclectic set of tool-mediated algorithms for particular cases, often using tools to avoid rather than confront the expression of particular properties. Perhaps it is only once learners have a sense of such a general method that they will become motivated to think about planes as mathematical rather than physical constructs.

The next chapter discusses how these various concerns were addressed in order that appropriate learning systems might be designed in which students' knowledge of the reflection transformation is brought closer to socially-accepted, institutionalised mathematical knowledge. The main research question guiding the design process was:

- *What knowledge, and in what forms, should be embedded into the expressive means and the instructional approaches of learning systems in order that students are supported in connecting the knowledge they have with the knowledge they are supposed to learn?*

The issues raised in this chapter have suggested the expressive means and instructional approaches by which learning systems are mediated have an impact in shaping the knowledge evolved in them. This raises a second research question:

- *To what extent do the different expressive means and instructional approaches incorporated in the learning systems constrain and/or afford actions and formalisations leading to evolutions in knowledge?*

  More specifically, what different meanings for, and analyses of, the transformation reflection, its properties and the objects upon which it operates evolve as the systems in which they are constructed evolve?

Chapter 4

Methodology of the Study

On teaching experiments......

> "There is perhaps no other type of research that more clearly illustrates the distinctive characteristics of research in mathematics and science education."
>
> (Kelley & Lesh, 2000; p.192)

> "It is a dynamic way of operating, serving a functional role in the lives of researchers as they strive to organise their activity to achieve their purposes and goals. In this, it is a living methodology designed initially for the exploration and explanation of students' mathematical activity."
>
> (Steffe & Thompson, 2000; p.274)

This chapter explains the methods by which the issues raised in the preceding chapters became operationalised and investigated as four learning systems, differentiated according to the instructional approaches filling-inwards and filling-outwards introduced in Chapter 2 and the use of microworlds based on either Logo and Cabri software considered in Chapter 3, were developed and evaluated. The various components that together constituted the learning systems were developed in parallel and in an iterative manner. This is to say that, rather than any one component being devised independently, tested in a pilot study and then used in a main study, the components were developed in parallel, with the evolution of each informed by and informing aspects of the others. This meant that the development of the systems formed a part of the study as important as the observations of the final versions in action. The research design adopted corresponds to a broad type of study that has recently been described under the heading of *teaching experiment* (see Kelley & Lesh, 2000).

The chapter is divided into three parts. The first presents the central themes and raises some overall methodological issues. The second section gives an overview of the learning systems developed and investigated, while the third summarises the data collection and analysis activities of the study.

4.1 Central themes and methodological issues

At the top level, the study attempts to understand how mathematical knowledge comes to be expressed when the means for its mediation varies. It asks questions about mediation by instructional approach and mediation by the available representation-systems.

As has been flagged in earlier discussions, placing mediation as a central concern has a number of methodological corollaries. Firstly, it means that it is not appropriate to consider the individual as agent without considering the mediational means with which he or she is operating. This is why Wertsch and Toma (1994) use the phrase *mediated agency* or *individual(s)-operating-with-mediational-means.* Second, it implies that the meaning of an action and the system within which it occurs are interconnected, which in turn indicates that action-in-system, and not individual learner, should be used as the unit of analysis (Cole and Wertsch, 1996). Following these ideas, it is possible to phrase the major aim of the study: it aims to investigate the ways in which knowledge about reflection comes to be expressed by groups of learners operating with different means, and how these expressions change as the individuals, the expressive means and the system in which they interact develop. The emphasis in this phrasing is the theme of *thinking-in-action.*

A third issue is raised by Noss and Hoyles (1996). They make the point that no-one can access directly the thinking of another, but that, if thinking is set in motion, it becomes more open to interpretation:

> "...we can set thinking in motion, and try to study what happens; we can set ideas in turbulence and investigate how changes occur; we can introduce new notions and try to understand how the thinker connects these with what she or he already knows."
>
> (Noss and Hoyles, 1996; p.9)

This suggests another way of phrasing the main aim of the study: it is a study of different ways of attempting to set thinking in motion. Here, the theme that is emphasised is that of *setting thinking in motion*.

Broadly speaking, these two themes correspond to two related aspects of the study.

- The design of learning systems intended to set into motion thinking related to the mathematical focus in question.

- The study of how thinking-in-action is contingent upon the various mediational means with which these systems are endowed.

In order to explore these themes in more detail, each in turn will be discussed below. However, it important to make clear that this separation is to some extent more a convenience than a true divide. Thinking-in-action can be expected to be shaped by the manner in which it is set into motion, but it is also the case that studying thinking-in-action can provide insight into ways of maintaining its momentum. This emphasises a process of *iterative design* that is appropriate when developing microworlds in which learners can express, explore and develop their own meanings for the embedded mathematical focus (see, diSessa, 1986; 1989; Pratt, 1998; and Stevenson 1996). With this is mind, the methodology adopted in this study will be described in terms of two phases: the *design phase* in which the theme emphasised was that of setting thinking in motion and the principle concern was on the development of microworld tools, sets of tasks and teaching interventions that characterised each of four different systems for exploring reflection; and the *comparison phase* which involved the analysis of detailed observation data collected when seven participants (the researcher and six students) brought each of the systems to life and hence concentrated on the theme of investigating thinking-in-action.

4.1.1 Teaching experiments as research design

The design of research related to the *teaching experiments* paradigm is discussed extensively in Kelly and Lesh, (2000). They describe a teaching experiment as:

> "..distinguishing itself by conscious breakdown of the researcher-teacher divide. The role of the researcher is recast, sometimes as a teacher, always as a co-learner. Similarly the roles of the students and teachers are often recast as cocollaborators in the search for critical issues, promising perspectives, relevant data or useful interpretations."
>
> (p.192)

The role of the researcher adopted in this study was consistent with this view. It was assumed that the shaping of (and by) the system through the researcher's participation is part and parcel of all aspects of the design and analysis. In the same vein, Angrosino and Mays de Pérez (2000) argue that any type of researcher observation maybe better rethought as a fundamental part of all systems for interaction. In this rethinking, it becomes appropriate for researchers to affirm and develop membership roles in the communities they investigate. This perspective can be used to describe the role of the researcher adopted in this study. The researcher became a part of the school community in which the research was conducted and acted as a member of each of the four learning systems, assuming the identity of teacher for the duration of the study.

Kelly and Lesh suggest that it is the type of research that happens in teaching experiments, above any other research design, that most clearly illustrates distinctive characteristics of research in mathematics and science education. In general, they are designed in order to focus on learning that occurs within conceptually rich environments specifically crafted to optimise the chances that development will not only occur, but occur in an observable form. Noss and Hoyles (1996) have argued that one of the *windows* opened by the presence of a computer offers just this, a methodological tool making more observable the thinking of those who use it.

> "The computer provides a screen on which learners can express their thinking, and simultaneously offers us the chance to glimpse the traces of their thought."
>
> (p.6)

So, teaching experiments are concerned with the investigation of evolutions in complex, self-organising, interacting systems, systems whose development (and whose components' development) is highly sensitive to small changes in conditions.

This raises questions about validity and replicability, which are considered in some detail by Lesh, Lovitts and Kelly (2000) and by Steffe and Thompson (2000). Steffe and Thompson also offer some reconsiderations on generalisability. Because every system studied in any teaching experiment is considered to some extent unique, they argue that

> "It does not make sense to demand of teaching experiments that they "generalize" in the way which one might hope that claims thought to be true about a random sample would be true as well about the population from which sample was drawn."
>
> (p.304)

The alternative view they offer is that the models of development resulting from researchers' analyses of teaching experiments serve generalisabilty to the extent that they might usefully be applied to systems beyond those that gave rise to them. They make an interesting connection between this view and the idea of generalisability in mathematics, where the pertinent issue is whether or not mathematical ideas used in one context also prove useful in others.

The analogy also provides a nice link to the idea of *situated abstraction*, a theoretical construct used by Hoyles and Noss to describe the process by which students "constructively generate mathematical ideas which are articulated in terms of the medium of construction" (Hoyles and Noss, 1993; p. 84). This would suggest that just as mathematical learning involves learners in identifying objects and relationships which resolve a particular problem *and* which might be usefully used in other domains as well, learning about mathematical learning involves developing ways of interpreting students' mathematical activity that explain not only the behaviour of participants in a particular teaching experiment but might be useful in explaining mathematical activity in systems beyond those of the original teaching experiment. It is this sense of generalisability to which the analyses throughout this project are aimed.

4.2 The learning systems

The four systems for learning which form the focus of analyses in this study were designed to provide arenas appropriate for Year 8 students (aged 12-13 years) to explore ideas related to geometrical transformation of reflection. Each system was constructed to include the same components:

- *Paper-and-pencil test*

  The first activity considered part of the learning system was a test designed to probe student views of, and competencies in, constructing, identifying and describing geometrical figures associated with reflection. The iterations through which the paper-and-pencil test passed informed other aspects in two ways: by enabling the production of a series of profiles of the conceptions of reflection among the sample from which participants in the systems were drawn; and by providing valuable data as to the kinds of tasks and teaching interventions that were appropriate for this group. The tests were administered in the last month of the school year to students from the four Year 7 classes who would be taught in the following year by the teachers who participated in the study.

- *Set of microworld tasks*

  The study aimed to develop a set of microworld tasks to be used in all four systems where the activities in each system could be regarded as far as possible as mathematically equivalent. The complete set (described in Chapter 5) included five computer-based activities to be completed in pairs. The overall aim of the activities was to engage students in building general expressions of reflection, using both visual and symbolic representation systems.

- *Final interview*

  The final activity for each system was an interview conducted with each student individually to build a picture of her views of reflection subsequent to microworld interaction.

- *Microworld kernels*

  Two different microworld kernels were designed. One a dynamic Euclidean geometry technical kernel (DEG), comprising a set of Cabri construction tools; and the other a multiple turtle geometry kernel (MTG), based on Logo (Microworlds Project Builder). The two microworlds both aimed to incorporate accessible means for constructing and expressing mathematical formalisations of transformation geometry, which would facilitate students in moving between visual and symbolic representations and between specific cases and general models.

- *Instructional approach*

  As well as designing the set of tasks and the microworld kernels, two parallel sets of intervention strategies were developed, one set to characterise the teacher-researcher mediations associated with a filling-outwards (FO) approach and the other with a filling-inwards (FI) approach.

- *Participants*

  The research was conducted in a girl-only comprehensive school in London; so all the participants involved in the study were female. The students (aged 11-13 years) were from mixed-ability Year 7 and 8 mathematics classes in which the SMILE (secondary mathematics independent learning environment) curriculum was followed. The organisation of this curriculum is such that each student works towards individually set targets. While this does not necessarily imply that all the students are working individually all the time and pair-work is common, at least in this school whole class teaching activities are infrequent. This classroom organisation allowed all the fieldwork related to the design of teaching tools and tasks in Phase 1 to be carried out in the normal classroom, with the work connected with the study integrated into the students' ongoing SMILE targets. In Phase 2, a different approach was adopted, and each group of six students worked with the researcher during their mathematics lesson but in a room separated from the rest of the class. Once again, however, the research work "counted" towards the students' targets, and hence was not viewed as extra work by the students. During both phases the class-teachers agreed not to set other work related to transformation geometry for the period of time during which students worked with the researcher.

Figure 4.1 schematises the organisation of components in each of the four systems for learning.

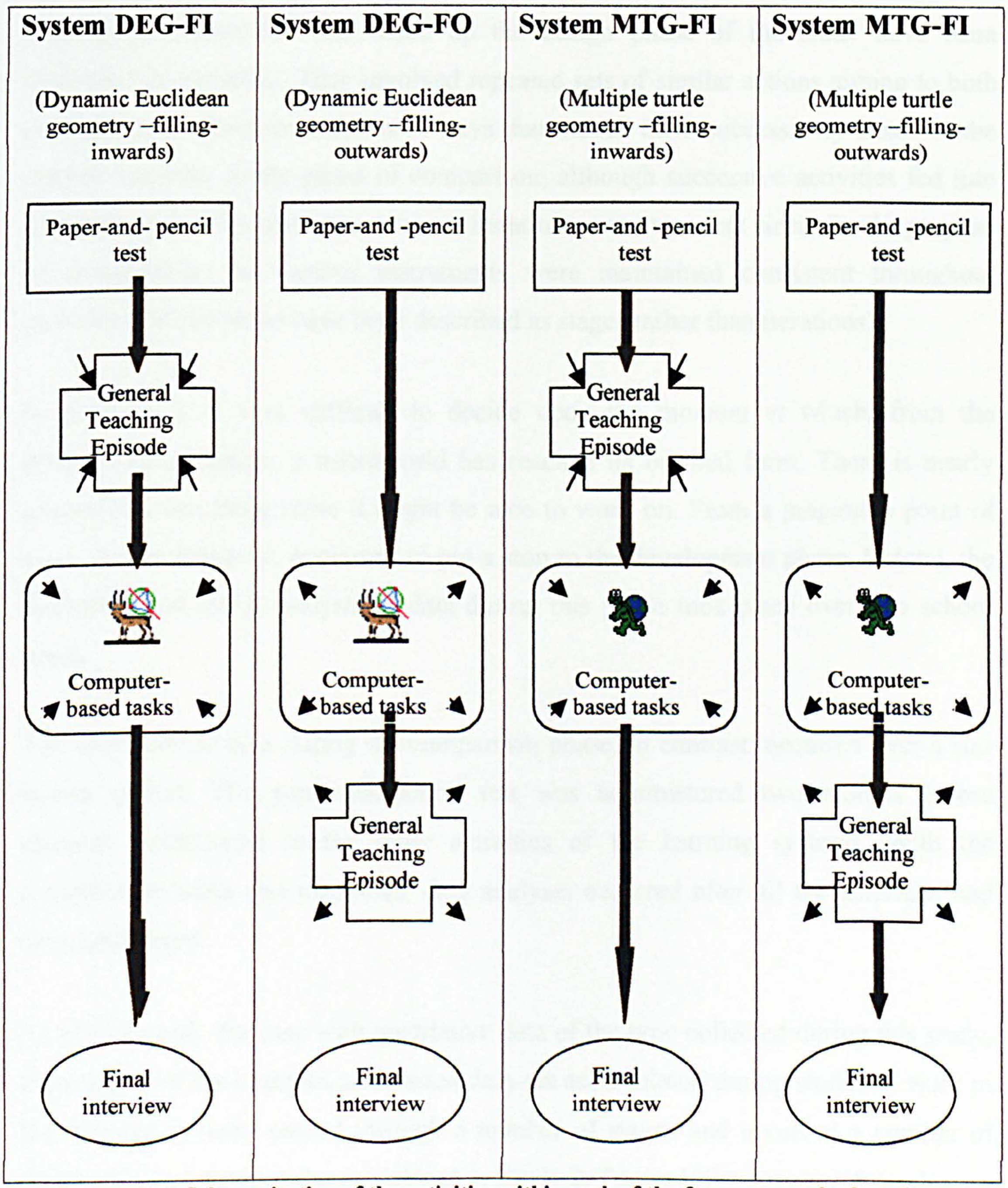


Figure 4.1: Schematisation of the activities within each of the four systems for learning

4.3 Overview of the research activities

As described above, the empirical work of the study was divided into two phases: the *design phase* associated with the theme of setting learners' thinking into action and the *comparison phase* which involved the analysis of such thinking-in-action.

The research activities that made up the design phase of the study have been described as iterations. They involved repeated sets of similar actions aiming to both constrain and afford interactions in ways that moved them successively closer to the desired patterns. In the phase of comparison, although successive activities fed into and from each other, the same sense of iteration was not present since, for the purpose of comparison, the various instruments were maintained consistent throughout (activities of this phase have been described as stages rather than iterations).

In practice, it is very difficult to decide upon the moment at which, from the perspective of design, a microworld has reached its optimal form. There is nearly always just one thing more it might be nice to work on. From a pragmatic point of view, it was, however, necessary to put a stop to the development phase. In total, the collection and initial analysis of data during this phase took place over two school years.

The collection of data during the comparison phase, in contrast, occurred over a six-month period. The paper-and-pencil test was administered two months before students participated in the other activities of the learning systems. With the exception of these test responses, data analysis occurred after all the activities had been completed.

As is commonly the case with qualitative data of the type collected during this study, the analysis of the complex and varied data-set accumulated during students' work in the learning systems passed through a number of stages and involved a number of different "cuts" through the data (Hoyles, Healy & Pozzi, 1994; Confrey & Lachance, 2000). In this process, the researcher strived to construct a coherent, consistent and comprehensible story of the development of the systems under study, in this case from the point of views of evolutions in the students' ideas about reflection (as evidenced in their activities and expressions), and how these ideas were mediated according to the tools and structuring of the learning systems. The cuts used in these study were driven both by theory and by the data, with some categories for organising the data constructed on the basis of considerations of the literature presented in the previous two chapters and others emerging from empirically-observed patterns in the data themselves.

Table 4.1 summarises the aims of the design phase and outlines the data collection and analysis measures used. Table 4.2 presents the same information for the comparison phase. The information presented in these tables is discussed in more detail in *§4.3.1* and *§4.3.2* respectively.

	Iteration 1 *Ground-laying*	Iteration 2 *Shaping tasks and tools*	Iteration 3 *Pedagogic structuring*
Aims	Probe students' "spontaneous" notions of reflection, transformation and plane geometry. Develop prototype tool-sets for DEG and MTG. Develop tasks to introduce tools of both microworlds with similar mathematical content.	Pilot paper-and-pencil test of students' views and competencies. Modify tool-sets. Select "good" tasks and organising into comparable sequences.	Develop teaching episodes to encourage abstraction and concretion. Pilot learning systems.
Data Collection	Transcripts of interviews with 23 students. (15 -30 mins). Researcher's notes, students' computer constructions and written work. (37 pairs of students working 45mins - 3hrs)	91 completed paper-and-pencil tests. Researcher's notes, students' computer constructions and written work . (33 students working individually or in pairs).	Researcher's notes, students' computer constructions and written work, partial transcripts. (4 pairs of students).
Data Analysis	Development of systems for classifying students' views of mathematical notions. Assessment of ease of use of tools. Identification of desired tools. Assessment of accessibility of tasks. Consideration of diversity of potential and actual strategies and solution.	Application and modification of classification systems for written responses. Assessment of ease and extent of use of tools. Identification of more desired tools. Assessment of accessibility of tasks. Consideration of diversity of potential and actual strategies and solutions.	Identification of teaching foci for teaching episodes. Development of ways of introducing tools and intervening during computer interaction. Ease and extent of use of tools.

Table 4.1: Overview of the aims and methodology in the design phase

	Stage 1 *Selection of students*	Stage 2a *Construction of case histories*	Stage 2b *Analysis of case histories*
Aims	Document views and competencies of student population. Select four groups with representative views and competencies.	Summarise the trajectory of individuals and groups through the learning system.	Identify conceptions and analyses of reflection and its objects as expressed by students.. Investigate differences student strategies associated with use of DEG or MTG microworld Examine impact of instructional approaches
Data Collection	88 completed paper-and-pencil tests	Researcher's notes, students' computer constructions and written work, along with transcripts of pair work on 5 computer-based tasks. (12 pairs of students spread across 4 groups, each task 90 minutes). Researcher's recollections, teaching materials and transcripts of general teaching episodes (4 groups of 6 students plus teacher-researcher, 45 minutes). Students' written work and transcripts of individual (24 students, 20-35 minutes).	Data used for analysis in stage 2b resulted from the analysis of data in stage 2a.
Data Analysis	Classification of students' views of reflection. Classification of student types..	Organisation of data into: initial profiles (according to test responses) process profiles and final profiles (based on activities in final interview)	Analysis of student strategies across systems, looking at differences in: epistemological considerations, representational means, microworld evolutions.

Table 4.2: Overview of the aims and methodology in the comparison phase

4.3.1 Iterations in the design phase

The design phase consisted of three iterations. Iteration one (*ground-laying*) was dominated by two parallel activities: (i) loosely-structured informal interviews with individual students to discuss their views of reflection, transformation geometry, figures and planes; (ii) building the first set of tools and associated tasks for each microworld. During this iteration, 23 students were interviewed, and a total of 37 students trialled activities in the prototype microworlds. The objective during this iteration was to gather data about students' views of the mathematical issues under investigation and to begin to develop systems to classify these views, as well as to ascertain the extent to which the tools were accessible and supported students in constructing formalisations which appeared to make sense to them.

During the second iteration (*piloting of tasks and tools*), the questions that had formed the basis for the interviews during the ground-laying iteration were redrafted into two versions of a paper-and-pencil test and activities for a final interview. The intention at this point was to have questionnaires matched to instructional approach. Two versions of questionnaires were administered, one version to forty-five students from two classes and the other to forty-six from another two classes. The focus of the analysis was on students' written descriptions of reflection and whether they matched those obtained during the interviews in the previous iteration and the aspects of task presentation that appeared to influence performance. The computer-based tasks and tools used in the previous iteration were redrafted, augmented and reorganised, based on observations of students' interactions with each other, with the microworld tools and with the researcher. A total of thirty-three students, variously working individually and/or in pairs were involved during this testing period. An important aspect of this iteration was to classify tasks according to their epistemological status and to sequence them in ways that permitted students to build upon the strategies and conceptualisations that appeared to make sense to them.

During the final iteration of the design phase (*pedagogic structuring*), the main focus was on the operationalisation of the two different instructional approaches. To the evolving task sequences was added a general teaching episode in which the mathematical concepts involved in understanding reflection as a transformation of the plane were discussed. Additionally, the presentation of each of the computer-based activities was structured according to instructional approach, with specific teaching episodes related to the task grafted before or after interaction. In both FI and FO approaches, students were introduced to a sub-set of microworld tools relevant to the task in hand before undertaking the computer-based activities.

This structuring was piloted as four pairs of students worked through the set of tasks as they had emerged after the second iteration. Two pairs followed a filling-outwards approach, one working with the DEG microworld and the other with MTG. Another two pairs worked to attempt the same tasks structured according to the filling-inwards approach. Similarly, one pair used the DEG microworld while the others interacted with the tools of the MTG microworld. Audio recordings were made of the interactions of the pairs with each other and with the researcher during the teaching episodes and the computer-based tasks. Pair-pair interactions were not analysed during this iteration, but partial transcripts were made in order to examine researcher interventions and to draw up a strategy for intervening in students' ongoing microworld interaction for each instructional approach.

4.3.1 Stages in the comparison phase

While the first phase concerned the design of the systems, the second involved their analysis in action. To bring the systems to life, four groups of six students were needed. The selection of the students was determined in the first activity of the second phase, which involved the administration and analysis of the *paper-and-pencil test.* The rest of this phase involved observation and analysis of each of the four selected student groups as they worked on the task-set and discussed their work with the researcher during the final interview.

4.3.2.1 Selecting the participants for the learning systems

In order to facilitate comparisons between groups, the ideal situation would be to have groups of individuals with identical conceptions about reflection, and identical histories in terms of their previous experiences (in and out of the mathematics classroom, with the transformation in question, with mathematics more generally and with working with each other). Of course, this is not possible. Each learner brings to any learning situation knowledge constructed as part of their unique trajectory through life. Hence, learners can have aspects in common but cannot be expected to be exactly the same. As the same individuals could not simultaneously participate in all four systems, what was needed was some way of constituting comparable groups, which could be considered reasonable representative of the larger population from which they are drawn.

The paper-and-pencil test was designed with this aim in mind. This test was administered to four classes of Year 7 students in the last month of the school year. The test was completed by students during a "double-period" of mathematics, that is a lesson of one hour and 30 minutes duration. It was presented by their class-teachers. To ensure that the students in all four classes received the same instructions, the procedure for administration was negotiated in discussions between the four teachers and the researcher. The teachers agreed to conduct the survey in a formal way and not to intervene or help the students in their work, although they asked that they be allowed to give some support to students who they felt would be unable to work independently. The scripts of this last group were removed from the rest of the data and were not analysed.

Before they began the test, the teachers explained to the students that it formed part of a research project and that the results were important to help the researcher understand their thinking about reflection. The teachers stressed that it was not a school test, the results would not be used as any part of the school assessments and that the students' responses would be kept confidential[1]. Students were asked to take as much time as they needed to complete the questions, and when they were sure they had finished to hand it in to the teacher and continue with their own class-work. As

[1] For this reason, all the names used in the reporting of data have been changed.

students worked on the test-items, they were allowed access to rulers, angle indicators, protractors and compasses but not to mirrors.

Completed scripts were collected from eighty-eight students in total. The student scripts were coded by the researcher and stored electronically. The classification systems used in this analysis were those developed during the design phase and described in some detail in Chapter 5. These classifications were used to organise students' responses according to: their views of the reflection and reflective symmetry; the nature of the sketches they produced and chose as representations of images under reflection or as axes of reflection; the general properties they recognise as invariant when asked to look beyond a particular figure; and their methods for constructing axis of reflection. These overall profiles gave a detailed picture of the conceptions held by the entire sample of girls.

The four groups of six students were constructed so that a range of views expressed and competencies in choosing and constructing representations of reflective symmetry in the two-dimensional plane was represented in each group (the selection criteria used are discussed in Chapter 6). The procedure for selection involved identifying from each class, six first-choice students and six reserves. The respective class-teachers were asked for their advice about the first-choice students, so that poor attenders could be excluded and the group composed in a way the teacher felt most conducive to individual and collective benefit. As a result of these discussions two groups remained unchanged, one change was made to the third group and two changes to the fourth.

Since the test had been carefully designed to include items that elicited interesting responses during phase one of the study, it would be inconsistent to consider students' responses to the questions as some kind of objective measure of their cognitive resources prior to participating with the computer-integrating task-sets. Despite the fact that students completed the tests individually, since they were administered in a mathematics classroom by their mathematics teacher, the situation also carried social connotations which can be expected to impinge on student responses (Schubauer-Leoni & Perret-Clermont, 1997; Lesh & Kelley, 2000). Moreover, the possibility that students' ideas about reflection were influenced by the items they completed, that is that they were learning as they completed the questionnaire, cannot be ruled out. To do so would contradict the idea that action-in-setting should be used as the unit of analysis. For this reason, the test is not considered as a pre-test illustrative of the knowledge of students before they participated in the study, but rather as the first in the sequence of activities addressing the mathematics of reflection. In this vein, results from the analysis of test responses are viewed as indicative of the ways students were provoked, at that moment, to mobilise a particular set of knowledge (not all mathematical) given the task demands and the mediational means available to them.

4.3.2.2 Observation and analysis of the systems-in-action

The final stage involved analysing the activities of the participants as they worked on the computer-based tasks and as they reflected on all their project-based activities in the final interview.

Data Collection

Computer-based tasks: All four learning systems included five sessions in which students worked on computer-based tasks. Each session lasted for 1 hour and 30 minutes. To tackle the microworld tasks, three computers were made available and the six students organised themselves into three pairs which remained constant throughout the five sessions. The computers were spaced at some distance from each other, not to exclude the possibility of between-pair communications, but to enable audible audio recordings to be simultaneously made, capturing the dialogue of all three pairs. These recordings were transcribed and along with written worksheets completed by the students during the five sessions, students' computer work and researcher notes made during and immediately following each session comprised the data collected during this activity type.

General teaching episodes: Because the researcher also assumed the role of teacher, it was not possible to make notes during the teaching episodes, although written descriptions were registered by the researcher immediately following these episodes. All the materials used and produced during these episodes were collected.

Final interview: During the final interview, students were given a task of constructing a paper-and-pencil image of a plane after reflection and were re-presented with their original paper-and-pencil test-scripts. They were particularly asked to comment upon their written descriptions of reflection and reflective symmetry. These interviews were audio-taped.

Data Analysis

As outlined above the aim of the data analysis activities was the production of a coherent, consistent and comprehensible story of the development of the four systems under study. The first step in the construction of such stories involved the researcher's written recollections made immediately following each of the six sessions (five computer-based tasks plus one general teaching episode). In these, the researcher recorded at a general level the strategies used by the students and the interventions that had been necessary; observations about the relative involvement of the students; aspects of the interactions that had seemed particularly positive and those that had more worrying overtones.

The production of these recollections accompanied the learning system activities. The long process of retrospective data analysis began with the construction of case histories for each of the twenty-four students who interacted in the learning systems. These case histories themselves had their own cyclical and iterative history, and were built up first by outlining the student profiles organised into three types: *initial profiles* according to their responses to the initial questionnaire; *process profiles*, complied from data collected during the computer-based tasks; and *final profiles*, constructed as a result of their productions and reflections in the final interviews.

The process profiles, in their first form, contained just the students' written or computer productions. Each profile was then expanded by sifting through the transcript data and the researcher notes to describe the strategies used during their construction. This organisation was informed by the issues emerging from the literature reviews in the previous two chapters and attempts were made to distinguish between moments of *empirical observation* and moments of *theoretical analysis*. Relevant extracts from students' discussions were isolated in the raw transcripts and copied and pasted into the case history, then the students' talk was annotated by meshing in data from the researcher's observations and including details of any relevant researcher interventions. In general, the same the process profiles were used to describe the work of both students in each pair.

The profiles were not constructed as finished documents; they represented the first level reorganisation of the data, and, at the second level, they themselves became the focus of analysis. Once again, this level was also characterised by a number of steps. The first investigation involved a consideration of *the strategies used in the computer-based tasks* and considered correct by the students. The tasks had been sequenced according to an epistemological analysis developed from the intra-inter-trans framework of Piaget and Garcia described in *§2.1.3*. Each strategy was considered according to: the mode of epistemological analysis that characterised it; whether it represented a general or particular solution; the properties of reflection that had been included explicitly in description of construction or validation activities, and the consideration of visual and symbolic aspects that had accompanied the evolution of the solution. In all cases, analysis sought to identify shifts in the ways that students were attending to reflection, its properties and its objects.

The next step was to analyse variations in the strategies adopted within and between the learning systems. The aim of this analysis was to assess the extent that differences in the internal resources of students and in the instructional approaches impacted on students' interactions with the microworlds. During this step, attention was also given at the between-system level to the researcher interventions that had been made.

4.4 Summary

This chapter has attempted to describe the methodology that underpinned the analyses that will be presented in the next two chapters. The theoretical issues presented in Chapters 2 and 3 emphasised the assumptions upon which the study is based, and principally, that mathematics learning is a process that takes place in self-regulating individuals interacting in self-regulating systems according to the resources they have available. The design retrospectively described as teaching experiment was chosen as most appropriate given these assumptions. The general methods used to organise the empirical work of the study were described in terms of two phases: design and comparison. The chapter has concentrated on outlining the methods used and has not presented the specific contents of the categories used for data organisation. Because of the iterative nature of the study, these classification systems emerged during the design phase and hence represent part and parcel of the results of this phase and are presented in the following chapter. In Chapter 6, they are applied in the analysis of the systems-in-action.

Chapter 5

Designing the four learning systems:

Tests, tools, tasks and teaching

> "... it's when...a shape cut in half, and one half is the reflection of the other, if you imagine a mirror in the middle, you would see the whole shape exactly the same on both sides."
>
> (Martha, 13 year-old mathematics student)

The previous chapter set out the methods adopted in the study overall, describing how it was broadly conducted through two phases. This chapter documents the various activities of Phase 1 that culminated in the versions of the tests, tools, tasks and teaching interventions that were to comprise the four learning systems adopted during Phase 2. It presents an analysis of the knowledge about reflection that students did and did not express in response to paper-and pencil-activities. It also describes how microworld activities were designed so that students might use their existing knowledge – their internal resources – together with external resources offered in the tools of the microworld and in the teaching episodes to develop and express mathematical meanings of reflection in ways that would correspond to its definition in socially recognised mathematical practices.

As well as providing the story of how the components of the learning systems co-evolved over time, this chapter also describes the different coding systems developed to make sense of the data collected. The organisation of these data, their analysis and the identification of emergent patterns, were all grounded both in theory as discussed in Chapters 2 and 3 as well as in the data themselves (see, Confrey & Larouche, 1998). The chapter has three main sections, each section corresponding to one of the three design iterations.

5.1 Iteration 1: *Ground laying*

The objectives of the first iteration were threefold: to probe students' existing knowledge of reflection, to develop prototype tool sets for two different microworlds and to design and test a set of reflection tasks. The research activities directed towards achieving these objectives were organised in parallel activities, as described in the following two sections.

5.1.1 Probing students' knowledge about reflection

In Chapter 3, research into students' performances on a range of transformation geometry-related activities was presented. This research provided a preliminary picture of how students might be expected to perform with particular types of tasks and the different ways they might conceptualise the objects of transformation (see *§3.3.3*). It was important to confirm whether the students involved in this study were reasonably typical in their responses.

To this end, in-depth interviews were conducted with individual students. An interview schedule was devised comprising three parts, each of which probed reflection from a different viewpoint: students' meanings for relevant mathematical terms related to the objects of reflection as discussed in *§3.4*; strategies used to classify figures according to their properties of reflective symmetry; the construction and validation of reflective images. The tasks that formed the focus for these interviews were based on those found in the mathematics curriculum followed in the research school.

5.1.1.1 Structure of the interviews

In the first part, the following mathematical terms were presented verbally to students: *transformation geometry, figures, planes, reflection* and *reflective symmetry*. They were asked if they had encountered these terms and in what contexts. If they described non-mathematical contexts, they were asked specifically if they had also come across the same terms in connection with mathematics and what they thought they meant.

In the second part, students were asked to classify four quadrilaterals. The researcher presented four statements about the number of axes of reflection (described to the students as lines of reflective symmetry) and the students were asked to match these statements with the corresponding quadrilateral (see Figure 5.1).

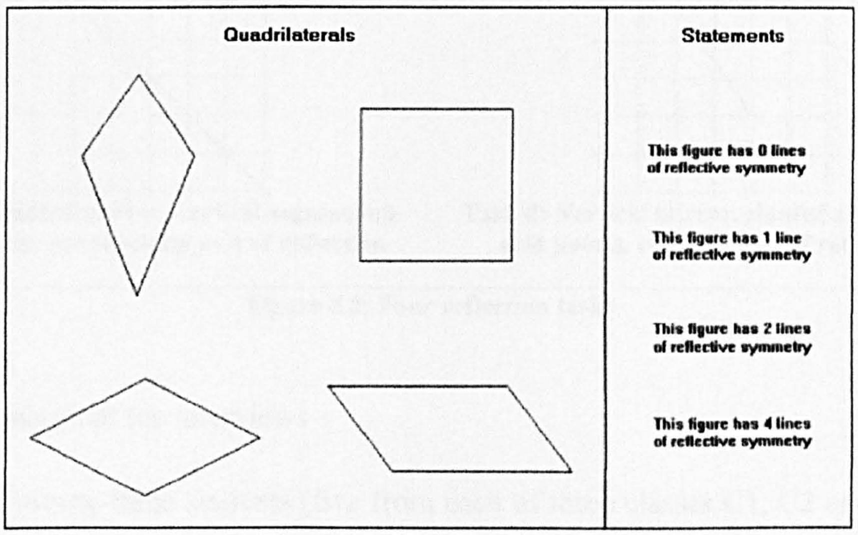
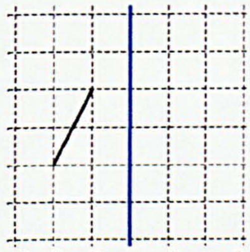
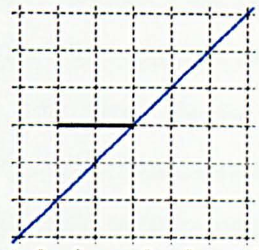


Figure 5.1: Classification task

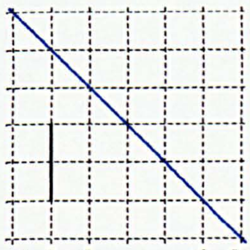
In the third part of the interview, the students were asked to construct the images under reflection in a given line of four line segments presented on squared paper (see Figure 5.2), and then to describe how they had decided where to place the images.



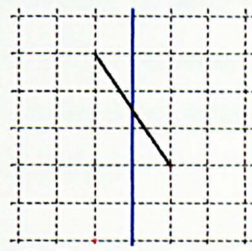
Task a: Vertical mirror, slanted segment on grid points, not touching



Task b: slanted mirror, horizontal segment on grid points, touching axis of reflection



Task c: Slanted mirror, vertical segment on grid points, not touching axis of reflection



Task d: Vertical mirror, slanted segment on grid points, crossing axis of reflection

Figure 5.2: Four reflection tasks

5.1.1.2 Analysis of the interviews

A total of twenty-three students (five from each of three classes C1, C2 and C4, and eight from the fourth, C3) were interviewed. The students were selected by the teachers in order to provide a rough cross-section of attainment levels of the class (although students with special difficulties were not included). Students were given access to rulers, compasses and angle protractors, although in fact only rulers were used. Mirrors were not available. Each interview lasted between 15 and 30 minutes and was transcribed.

The following three sections describe the organisation of student responses to each of the three interview activities. The emphasis in these descriptions will be on the development of appropriate ways to code the qualitative data obtained and how these related to the theoretical constructs introduced in at the end of Chapter 3 (see *§3.4*).

Part 1: Students' descriptions of the mathematical terms

In this section, regularities in students' descriptions of the mathematical terms are identified.

The term '*transformation geometry*' was not familiar to the students and no-one was able to describe the term with any clarity. Neither were any students able to connect the term '*plane*' with any mathematical content. The term '*figure*' had various meanings for the students and was associated with both mathematical and non-mathematical contexts. Specific examples in the geometry context included various polygons (usually types of triangles or quadrilaterals) and circles. Students did not talk of lines, line-segments or points, but described figures in their entirety. After consideration of student responses, it was decided it would not be productive to solicit written descriptions of these three terms: the first two were too unfamiliar and the third could be assessed from students' responses to other items.

Reflection and reflective symmetry: All twenty-three students were familiar with the term '*reflection*' and reported having come across it in their mathematics lessons. Six students said they had never heard of '*reflective symmetry*', although three of these students said they knew the term '*symmetry*'. Students' responses were first grouped according to frequently emerging references, with five groups identified. These were references to: *mirrors* or other reflective surfaces (20 students), *behaviour of light* (5 students), *congruency* (19 students), *reversal in orientation* (11 students) and *divisions of the working space* (9 students). Examples associated with these groups are presented below. Descriptors are not exclusive and often the same expression included references to more than one group.

Comments about mirrors and about the behaviour of light were commonly related to the action of 'doing' a reflection, as can be seen in the comments of Jane and Rita below.

> *"Reflection is when you use a mirror to reflect something and you can draw its reflection on the other side."*
>
> (Jane, C2).

> *"Reflections are when a source of light is bounced off a reflective surface and produces an identical image of the object that appears to be behind the surface".*
>
> (Rita, C3)

Although the idea of light rebounding back from a reflective surface might evoke visual images of equal angles, Rita's description was representative in that no mathematical properties of the 'bouncing' activity of light were expressed. Division of space was also mentioned in both the above two comments: Rita refered to a kind of virtual space behind a surface, while Jane, although she is not explicit, seemed to be describing "the other side" as a physical space in which drawings can be made.

The references classified as relating to congruency and reversal in orientation centred on relationships between pre-image and image, as illustrated in the following two comments:

> *"Reflection and reflective symmetry, I think they are the same thing, you have two halves that are identical ... half a face and then the reflection is the other half and the face is symmetrical."*
>
> (Carla, C4)

> *"I would describe it, that in reflection, it's when you have a picture and a mirror and on the other side of the mirror you make the same picture, only back to front."*
>
> (Neema, C3)

The idea that the reflective image is congruent to the pre-image was present in the responses of both Carla and Neema. They differ in that Carla spoke of one object and its internal relations, while Neema seemed to imagine two objects and tried to describe the relationship between them. It is tempting to classify this difference in terms of the categories of intrafigural and interfigural defined in *§2.1.3*. However, although relationships within single figures characterise intrafigural analyses, Piaget and Garcia add that comparisons between the internal properties of two or more figures should also be considered as intrafigural (1989; p.113). It is when attention is given to the position of figures within a surrounding space that interfigural relationships come into play. This would suggest that the observation that two figures are congruent should be classified as intra rather than interfigural unless the process by which one has been transformed to produce the other is also discussed.

The reference to reversal is perhaps more ambiguous, since the phrase "back to front" appears to evoke relative position in space. Yet there is still no allusion to any transformation that would associate the initial picture with its final image, of any geometrical properties that underlie their relative positions. In this sense, focus still seems to be on intra- and not inter- relationships.

Perhaps the intra-inter distinction can be made clearer by considering Piaget and Garcia's classification of geometrical locations. The construction of a midpoint does not necessarily require the organisation of space (in this case, a two-dimensional plane) as a structured totality and hence can be completed at the intrafigural level. The construction of the locus of all the points equidistant to two others, on the other hand, pertains to the interfigural. It involves abstracting a relationship between the two points and all the members of the set of points that make up the solution.

In short, neither Carla nor Neema – nor any other of the twenty-three students – provided descriptions that mathematised the space between the figures. The only explicit treatment of space involved the idea that it could be divided by a line or by a surface into 'sides'. So, although students did know that any pre-image/image pair cannot be located on the same side of the axis of reflection, focus on the properties of reflection was almost exclusively associated with intrafigural interpretations. This suggests any teaching sequence should include activities and interventions designed to encourage students to integrate interfigural perspectives into their problem-solving strategies.

As well as organising students' responses in terms of frequently emerging references, they were also classified in terms of the three different meanings that students seemed to associate with reflection: *physical process*, a *perceived object* and a *unary property*. There was some overlap between the two classifications systems: for example, when reflection was used as a verb to describe physical processes, references to mirrors and the behaviour of light were often made (see Jane's comment above and Rachel's below).

"When a mirror is reflecting, it makes an image that appears on the other side. So what you do is, you take the mirror and put it on the line of symmetry and you draw the exact copy of the other side of the paper."

(Rachel, C4)

What is absent from descriptions like Rachel's is explicit reference to the mathematical, or even physical, invariances associated with this process, rather than its product. This offers evidence to support the conjecture first mentioned in *§3.3.3.3*, that use of tools that take care of the process of transforming leads to views emphasising their products – and, again, privilege intrafigural over interfigural concerns.

Students also used the word 'reflection' as a noun. It often, as was the case for both Jane and Carla, signified image as a *perceived object*. This is quite different from reflection as a functional object (see Grenier & Laborde, 1988; p. 66) that can be applied to map the plane onto itself. The use of reflection to signify image also privileged intrafigural relations, especially since students tended to describe pre-image and image, but not their relationship with the axis of reflection.

The third meaning located in students' talk is that of reflection/reflective symmetry as a property or relationship between two objects or between two parts of the same object (this is evident, for example, in the excerpt from the interview with Carla quoted above). Looking across all the interviews, the property, in Carla's case described as symmetrical, represented a unary predicate which was applied to the complete configuration, usually a single figure (see also, Vergnaud, 1997). None of the students talked about two figures being symmetrical in relation to any axis of reflection. In fact, none of the students interviewed spontaneously mentioned the relationship between distance between an object, an axis of reflection and the image of the object, although, as will be described later, this was clearly something they used when drawing reflections.

The categories presented above served to organise student data and were used in future iterations to classify written responses to a paper-and-pencil test. By considering more extended discussions between the student and researcher, the abstraction and concretion processes evident in students' work that would inform the analysis, not of test items, but of the students' interactions within the learning systems, were identified. This involved locating movements between what can be described as empirical analysis (in the sense of originating in or based on observation or experience) and theoretical approaches. The following extract from Martha's interview shows how she moved from apparently empirically motivated abstractions of the effects of reflective surfaces, to a more theoretically driven concretion in the mathematical context.

Mar: *A reflection is caused by something that reflects. A mirror, or a river, or a shiny...anything shiny, and you see the image of an object being reflected.*

Res: Yes...and do you know anything about the image?

Mar: *It's the same as...well, I'm not sure, it depends... it's not always the same.*

Res: Not always the same?

Mar: *I think it depends on the surface, in water it can be all wobbly.*

Res: And can it be wobbly in a mirror?

Mar: *...If it's a wobbly mirror, I suppose (laughs).*

Res: Yes, I suppose...what about reflective symmetry?

Mar: *Mmm, it's when...a shape cut in half, and one half is the reflection of the other, if you imagine a mirror in the middle, you would see the whole shape exactly the same on both sides.*

Res: A wobbly mirror?

Mar: *No, it wouldn't work with a wobbly mirror.*

(Martha, C3)

Martha began by using reflection to signify image, and then included the idea of the physical process of reflecting, and finally described reflective symmetry as a property. In the beginning of the conversation, Martha seemed to associate reflections with the production of congruent objects, but as she thought of her experience with reflective surfaces outside the mathematics classroom, she recalled situations when the image was a distorted version of the original object. The term reflective symmetry brought Martha back to the mathematics context and to the idea of congruency.

This exchange is interesting from a number of points of view. First, Martha, associated reflections with mirrors inside and outside of the mathematics classroom. In what could be described as a process of abstraction, she described mirrors as elements of a larger collection of surfaces that reflect. When considering the term reflective symmetry, the context for Martha was more clearly limited to mathematics and the notion of tools that reflect is concretised, particularised to the group of non-wobbly mirrors that preserve congruency. It is important to note that there is nothing wrong or misconceived about Martha's thinking. If her comment about reflective images not always being the same size was considered in isolation, it might be concluded that she was unaware that an object and its reflective image are congruent. In her case, the other information she provides suggests that she does know this, but she also knows that, outside the mathematics context, reflection does not always have this meaning.

Rather than abstracting mathematical invariance from her experience with her lived-in world, Martha seemed to be able to use her knowledge about reflection in connection to mathematics to interpret which experiences of this world might be considered valid illustrations of a mathematical idea. This is rather different from abstracting relationships from empirical activity and could be interpreted as evidence of a reverse process in which theoretical ideas about mathematical invariances are concretised through empirical references.

Classifying quadrilaterals

In response to the task presented in the second part of the interview, the majority of students experienced difficulties in identifying the quadrilateral with no lines of symmetry. Only five immediately classified the parallelogram correctly. Of other students, some divided the parallelogram using its diagonals, some wanted to halve the quadrilateral by joining midpoints of opposite sides, and some wanted to do both (in which case they wanted to assign the property of four symmetrical lines). Since the interviewer repeatedly made clear that one figure had no lines of symmetry, the students struggled to identify their mistakes. Three of them realised that the lines joining the midpoints of the parallelogram were not lines of symmetry after adding these lines to the figures.

These three students seemed to be alerted by visual concerns: they *saw* that the vertices were not opposite and were able to assess, more or less, where each should be placed. The remaining students, however, seemed to have no way of validating their constructions. What they saw may have been a figure divided into two congruent parts, which satisfied at least their articulable knowledge about reflective symmetry. They were seeing a symmetrical figure, but a figure with rotational symmetry with respect to its centre and not reflective symmetry. Three gave up. The rest did go on to accept that the parallelogram had no lines of symmetry only after the researcher suggested folding.

This task was successful in creating a perturbation, but the lack of a non-physical means of validation meant that the students had difficulty in resolving this perturbation and the majority were not provoked to move to the interfigural relations that could have clarified the problem. From this analysis, it seems that, given problems involving the construction of axes of reflection, students are likely continue concentrating on intrafigural properties of figures.

Part 3: Drawing reflections

The items used in the third activity of the interviews were drawn from previous work (see Hoyles & Healy, 1997, Küchemann, 1981, Grenier, 1988). It was expected that responses would be associated with the following task features: orientation of mirror, orientation of the initial segment and position of segment relative to the mirror. The responses generally followed the patterns identified in previous research, although students' descriptions of their activities opened some new windows onto the possible motivations underlying the images produced.

The first item was completed with little difficulty, all the students drawing the image as shown in Figure 5.3.

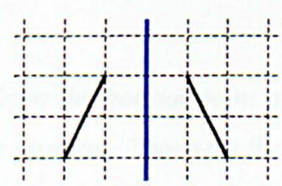


Figure 5.3: Applying a reflection in a vertical axis

For the second item, it was expected that some students would err by drawing the segment in the same horizontal orientation on the right side of the axis of reflection (shown in Figure 5.4).

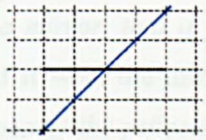


Figure 5.4: Constructing a horizontal image of a horizontal segment

Ten students initially produced this construction, although four of them went on to modify the figure later in the interview. Of the seventeen students who constructed the mathematically correct image, two counted across the diagonals of the squares (i.e., used a perpendicular relationship). Six other students described a different counting strategy, which they had used to position the point of the segment that was not on the axis of reflection:

"Starting from this point, I went one two across, and then one, two down."

(Kayleigh, C1)

Both these strategies involve some interfigural concerns, one treating the axis as a perpendicular bisector and the other as a bisector of the angle made at the point where a line meets its image, although Kayleigh's way of counting squares only ensures the angles each side are the same because of the 45° orientation of the axis. The image could also be constructed using intrafigural relations only and six students appeared to attempt to visualise a final configuration that the axis would "cut in half".

For the third item, twelve students constructed the image segment to be vertical, with three different locations chosen for the image (Figures 5.5a, response of seven students, 5.6a, sketched by three students and 5.7a, the production of two students). Below, part of the interview with one of the students whose drawing corresponded to that in Figure 5.5a is presented.

Gem: *I counted across from the bottom point and it was 4, so I counted 4 more and drew the segment, 2 squares there, 2 squares here.*

Res: And are you happy with the result?

Gem: *I think it's alright ... is it right?*

Res: How could you check?

Gem: I could use a mirror, put a mirror on this line.

Res: Uh huh... But we haven't got a mirror... Could you tell me, exactly how would you put it? How are you thinking of this line?

Gem: I think of it like it's a mirror, that if you put a mirror on this line, (places her hand as if it were the mirror along the axis at an angle perpendicular to the paper) *the reflection is on the other side* (waves her other hand in the space above the paper on the opposite side from the segment).

(Gemma, C2)

On paper, it looked as if Gemma had slid the complete segment horizontally across the page. This could be interpreted as a failure to take into account the orientation of the axis, or of the orientation of the segment in relation to the axis. Gemma thought the image she had constructed looked "alright", although she would have liked the interviewer's confirmation. When she was asked to think of her own validation method, she suggested the use of the mirror. This tool was not available, but as Gemma enacted the placing of the mirror onto the axis of reflection, another possible interpretation of her solution emerged. If the figure is revisioned as a two-dimensional representation of a three-dimensional situation[1] (Figure 5.5b), then it is no longer straightforward to classify the response as correct or incorrect. As Figures 5.6b and 5.7b show, it is also possible to revision the other two locations for the vertical images in this way. Similarly, it is possible to imagine a legitimate 3-D representation of the solution for the previous item presented above in Figure 5.4).

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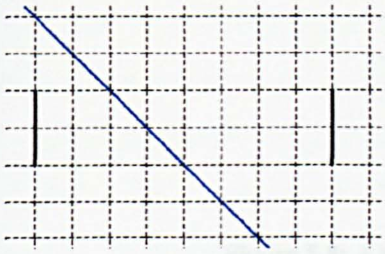
[1] I am grateful to Peter Winbourne, who first suggested this possible interpretation to me.



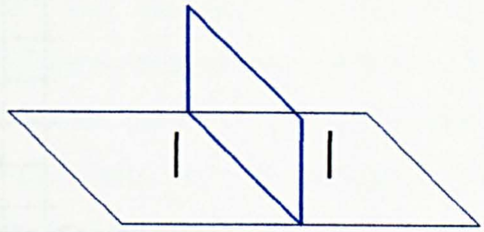
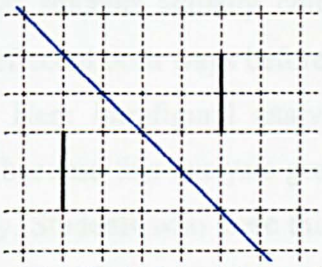


Figure 5.5a A horizontal sliding strategy? ... Figure 5.5b ... Or a possible 3-D interpretation



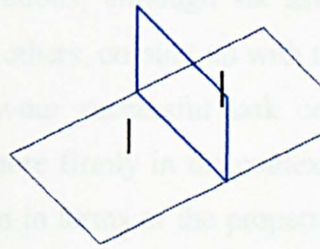
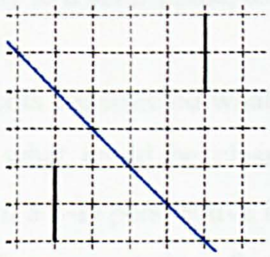


Figure 5.6a: Confusion with rotation? ... Figure 5.6b: ...Or a possible 3-D interpretation



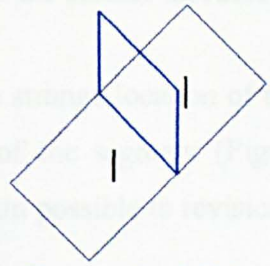


Figure 5.7a: Placing only one vertex correctly? Figure 5.7b: ...Or a possible 3-D interpretation

It would be misleading to conclude on the basis of Gemma's evocation of the physical mirror that all twelve students were actually imagining the three-dimensional case as they drew the vertical image. Their actions with the resources and their paper-and-pencil representation did not offer a clear window onto their mental representation. It is not even clear that Gemma's production was necessarily driven by a three-dimensional vision. The presence of the grid could be argued as evidence against such a vision, as it too really ought also be in perspective in the 3-D case. Nonetheless, it is a reasonable interpretation that students who attempt to connect reflections in the mathematics classroom with their everyday interactions with mirrors experience some confusion about whether they are working with 2-D or 3-D space.

Moving to the final item, it was expected that students might construct images in which only part of the segment on one side of the axis was transformed (e.g. see Figure 5.8).

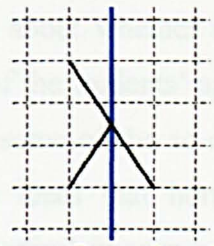


Figure 5.8: A transformation of a semi-plane

Only five students actually constructed such solutions, although six asked if the mirror reflected both ways before, along with nine others, coming up with the correct image. Here intrafigural analysis seemed to favour successful task completion, perhaps because this analysis grounded students more firmly in the context of plane geometry. Students who were thinking of reflection in terms of the properties of two parts of a final figure did not seem limited to thinking of reflection as a process that applies only to a semi-plane, as was reported in some of the studies discussed in *§3.3*.

Four students constructed what at first seemed rather a strange location of the image, based on what could be classified as a translation of the segment (Figure 5.9a). However, if a 3-D perspective is adopted, it is once again possible to revision this as a correct 2-D representation of a 3-D situation.

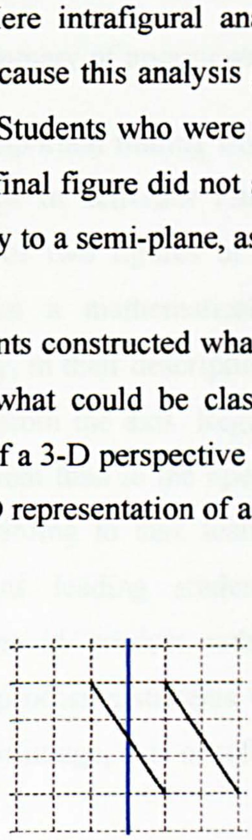


Figure 5.9a: Translating the segment?

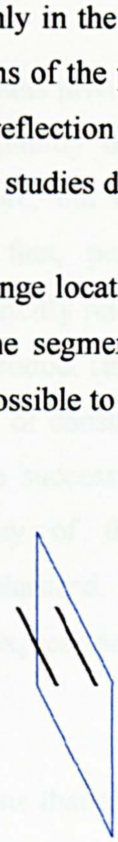


Figure 5.9b: ...Or a possible 3-D interpretation

In summary, the variables associated with students' responses to the paper-and-pencil based interview tasks match those identified in previous research and include: the orientation of the mirror and of the figures in the plane represented, and the presence of a grid which makes possible specific counting strategies. However, in attempting to make sense of students' actions and their descriptions of what they were doing, a possible confusion about the space in which reflections are applied has been identified.

It may be that a lack of clarity about whether they are working in two or three dimensions contributed to some of the students' apparently incorrect responses. This would suggest a need to rethink some of the so-called misconceptions identified in the literature. For example, the ideas that horizontal and vertical objects have correspondingly horizontal and vertical images or that, for a given object and axis, there can be more than one correct image (Bell, 1993 p. 131-132) can no longer be considered as necessarily misconceived.

5.1.1.3 Summary of interviews

The first important finding from the interviews is that students privileged intrafigural relationships in activities related to reflection. They explicitly described internal properties of two figures or two parts of the same figure, but not their relative positions in a mathematically structurable space. In fact, perhaps somewhat surprisingly, in their descriptions students did not even explicitly refer to the distance of objects from the axis. Regularities in the final visual product (the drawing) were more apparent than in the operations used in the processes of construction and these varied according to task features. Even when tasks were successful in provoking perturbations leading students to question the validity of their approaches, relationships in product rather than process were emphasised, as the physical validation processes students wanted to use avoided any expression of relationships between pre-image, axis of reflection and image.

It was only when students were required to enact reflections that the space between figures was given any attention. In practice, as Grenier (1988) concluded, this boiled down to some construction of equal distances (p.398) using essentially intrafigural relationships.

Analysis of students' activities during the interviews also resulted in the revisioning of some of students' seemingly incorrect constructions as correct two-dimensional representation of three-dimensional situations. This is to say, that if mirrors (physical objects with length, breadth and height) are presented as the main external resource for concretising the mathematics of reflection, students may quite reasonably associate the transformation with a set of situations quite different to those of its intended mathematical meaning. This would suggest that they might benefit from new resources that would afford them to construct and validate reflections in a world more readily mathematically constrained, in that the rules constraining activity are explainable and justifiable. In the absence of such a resource, one might hypothesise that the passage between intra and interfigural relationships is obscured by a corresponding tendency to move from 2-D to 3-D interpretations. For the task sequence, this suggests a need for tools that limit activity to the 2-D plane (in addition to mirrors, this also rules out folding).

When analysing figures from an intrafigural perspective it makes sense, in abstracting and concretising activities, to treat figures in their entirety or as collections of special defining points and line segments and it is not necessary to think of infinite sets of points. It is also quite possible to work at the intrafigural level with a rather vague notion of space as the place in which figures exist.

Students' intrafigural view of figure acted as an internal concretising resource, which could constrain the consideration of reflection to a mathematical domain. It could also be used as a resource for making abstractions, enabling the identification of general geometrical properties that describe internal relationships between the pre-image and its image under reflection. The intrafigural analysis hence represents an important, and seemingly accessible, part of the mathematical knowledge associated with this transformation. But it is only a part. The question is, can students build from an analysis of (two-dimensional) properties within figures an analysis that also takes account the properties of the (two-dimensional) space between them?

Using the tools and tasks of the paper-and-pencil setting (which also included angle indicators, rulers and compasses) of the interview, this movement was not made by the majority. So what tools and task might serve such a purpose? The option adopted in this study was to attempt to design computational environments in which the only way to manipulate and reconstruct objects is to express explicitly the relationships between them.

This comprised another activity of the ground-laying iteration that will be addressed in the following section. Before moving on to this discussion, the following quote highlights a critical methodological point associated with the design of tools for mathematical expression; they should allow just enough freedom of expression:

> "There is an important methodological point here. In order to study the forms of expression employed by learners to mediate ideas that closely conform to that which is deemed mathematical, we must tread a careful path between allowing free range to that expression and constraining it too tightly. In the former case, it might be difficult to capture any traces of the learner's thinking — let alone any traces that could be used as resources for mathematical learning. The latter approach runs the risk of constraining to the point of predictability, in which case we would merely be studying our own preferences. As researchers, let alone educators, we have to respect *and* constrain diversity."
>
> (Noss, Healy and Hoyles, 1997; p.213)

5.1.2 Tool and task development

At the same time that the interviews were being conducted, work was underway on constructing the tools-sets for the microworld kernels that would be used in the study. Neither microworld was to be built entirely from scratch: Cabri-géomètre software[2] was used for the dynamic Euclidean geometry (DEG) kernel and the multiple turtle geometry (MTG) microworld was based on a Logo microworld, Turtles Mirrors (see, Hoyles and Healy, 1997), written in the Microworlds Project Builder (LCSI, 1992).

At this point, the specific aims of the teaching experiment can be re-stated: that is, that in the course of the activities within the learning-systems, students' knowledge will evolve in ways that move them closer to mathematically significant knowledge about reflection as a geometric transformation. From the analyses already presented, this evolution will involve: extending mathematical analyses from intrafigural into

[2] Specifically, Cabri I version 2.1 was used.

interfigural[3] perspectives; and adding to views of transformation as a property or relationship of a geometrical configuration, the view of a transformation as a function or mapping of one geometrical configuration onto another configuration. Evolution is hence not conceived as suppressing one interpretation in favour of another, but as an extension of different interrelated knowledge.

In this first iteration, attention was primarily focussed on building accessible tools to support representation of both intra and interfigural relationships and engaging tasks by which students could explore these relationships. To test and refine the microworld tools developed, students worked in pairs or individually, with the researcher noting the extent to which the tools under construction matched what they wanted to do and enabled them to see beyond any particularities of their approaches to more general methods. In the following two sections, the tool-sets of the microworlds are presented and important milestones in their development discussed.

5.2.1.1 The dynamic Euclidean geometry tools

The choice of a computational environment that has as its model traditional Euclidean geometry might be thought as inappropriate to encourage interfigural considerations. After all, Piaget and Garcia in their historical analysis defined the period associated with Euclid as the '*intrafigural* period' characterised by a lack of consideration of space (see *§2.1.3*). However, in Cabri-géomètre, dependence relations between basic Euclidean elements and constructions can be described in terms of functions (Laborde, 1993 p.58). For example, using a line-segment as input, the midpoint construction outputs a point in the appropriate location whose subsequent movement is relative to the manipulation of the line-segment on which it depends (see also, Pratt & Ainley, 1996/7). The functional aspect emphasises transformations of figures within a space and can therefore be aligned with interfigural interpretations.

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[3] It also involves transfigural considerations, but since this moves one beyond the analysis of a particular transformation to its relationships to others, transfigural concerns will not represent a central focus in this study.

The first version of Cabri-géomètre was used in this study. This software was available on the school machines and also has a number of specific features making it suitable for the DEG kernel. One transformation tool was present in the default construction menu of the software: `symmetrical point`, which produces the image of a point of reflection in a given line or of a rotation of 180° about a given point. Because the fusion of symbolic with visual terms has been highlighted as an important concern of this study, the Cabri version selected for use was one that contained an `exposition` tool. This tool could be used to display a symbolic description corresponding to the menu selections and mouse clicks of the user.

The first DEG kernel

In addition to the `symmetric point` construction, a number of other constructions are also available in the default configuration of Cabri I[4]. None of these tools could easily be used to construct directly the intrafigural relationships that seem to characterise reflection to the students who would be using them. Some more construction tools, or macros, were hence added. This included two tools for constructing equal lengths, `compass(2)` and `compass(3)`. The `compass(2)` tool was added after a number of students experienced difficulties in constructing two equal lengths emanating from the same point (this requires only two inputs and the original `compass` tool needed three[5]). The `circle` tool could be used for this purpose, but the students tended to associate the function of this tool exclusively with the production of circles.

The difficulties students have in seeing a circle as a tool for preserving lengths has been noted in other studies (see for example, Jahn, 1998; Noss, Hoyles, Healy, and

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[4] The default construction tools are: `locus of point`, `point on object`, `intersection`, `midpoint`, `perpendicular bisector`, `parallel line`, `perpendicular line`, `centre of circle`, `symmetrical point` and `bisector`. The creation tools available are `basic point`, `basic line`, `basic circle`, `line-segment`, `line by 2 points`, `triangle` and `circle by centre & rad. point`.

[5] Re-using points within a macro is not possible in Cabri (an issue considered in some detail in Healy and Hoyles, in press).

Hölzl, 1994). Jahn (1998) suggests that it may be associated with a conceptualisation of circumference not as the set of points equidistant from a centre point, but as a closed curve with constant curvature (Artigue and Robinet, 1982).

To attempt to side-step the issues related to seeing the circle as a set of points rather than confront it, two versions of the compass tool were made available in the DEG kernel. `compass(2)` was, in function and in definition, the same as the original Cabri `circle by centre and rad. point` tool, but it was located together with the other `compass` tool under the construction menu and this seemed to help students accept its role in creating equal lengths. The name and location in the menu of the tools were hence important in framing students' operations with them, indicating the sensitivity of learners to apparently small details. In addition to the length copying tools, a set of tools for constructing equal angles was also included in the DEG kernel, `angle carry(3)`, `angle carry(4)` and `angle carry(5)`, the number in brackets again signifying the number of points to be used as inputs.

Having decided on the kernel, the next question was how to introduce students to it. The challenge was to introduce the critical difference between *creations* (drawings) that mess-up under dragging and *constructions* (figures) in which defining properties are preserved when elements are moved around the screen. For this study, a strategy similar to that used by Healy and Hoyles (in press) was adopted, whereby after a brief initial exploration of the creation menu and the dragging facility, students were introduced to a limited number of construction tools, which could be used to resolve a particular challenge that followed. To satisfy the principle of affording diversity, it was important that any challenge could be resolved in a variety of ways, even given the limited tool-set presented.

In the first attempt at an introductory activity, no mention of reflection was made and instead the teaching emphasis was entirely on the use of the new construction tools. The task devised involved the construction of a figure representing a stick-person, which could be moved in various ways. The figure was given to students in its complete form and their task was to produce an identical figure. The stick-person and its underlying construction are shown in Figure 5.10.

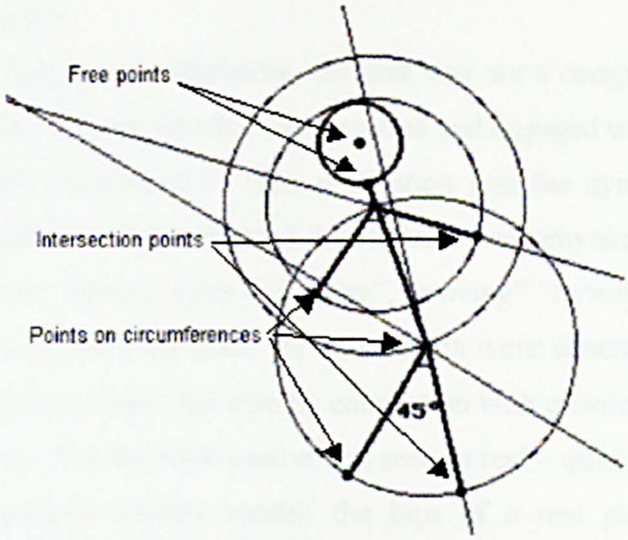

Figure 5.10: The Cabri stick-person construction

The activity began with a demonstration by the researcher in which the `compass`, `angle carry`, `point on object` and `intersection` tools were used to produce part of a congruent figure (Figure 5.11).

From the interactions of student pairs on this task, it became clear that it was too complex as an introduction. Nonetheless, some of the features of the tasks proved successful in engaging students in geometrical investigations and these favourable aspects are described below.

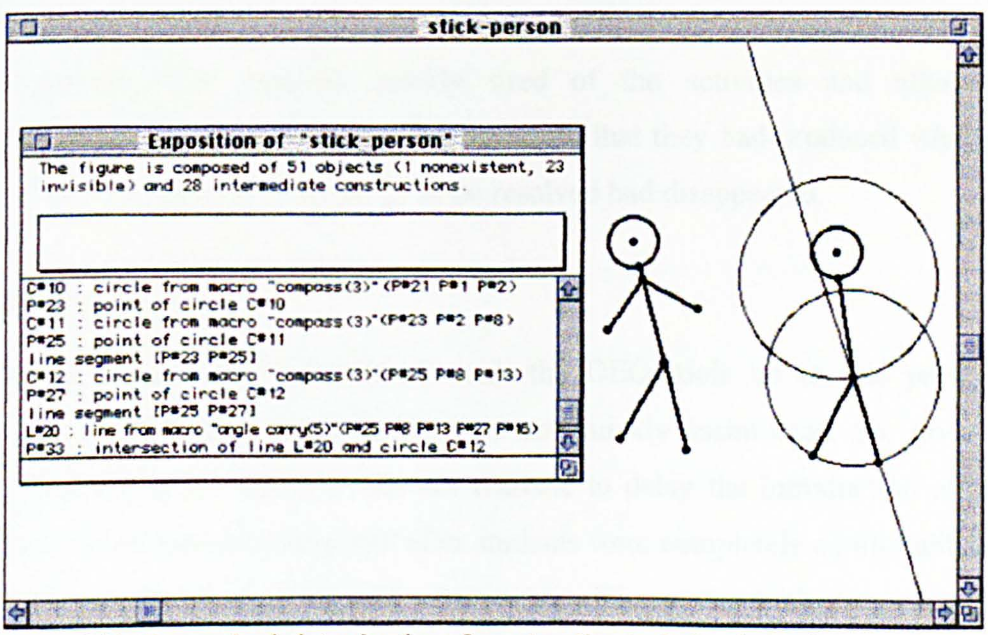

Figure 5.11: Researcher's introduction of construction tools (as shown in `exposition`)

- Favourable aspects

  Engagement: Despite its complexity, the task was not a complete disaster. All of the student-pairs understood what was required and engaged with the activity. The main factor that contributed to their motivation was the dynamic aspect of the figure – its animation also animated the students and provoked them to describe its actions in very human terms (*"dancing"*, *"waving"* *"turning head-over-heals"* etc.) as well as geometrical ones. Its movements were concretisable in terms of known geometrical objects, but also by connection with movements of real human bodies. It was not that the stick-person was seen as real – quite the contrary, it was clearly an extremely limited model: the hips of a real person hardly rotate following the trajectory of a perfect circle. But the fact that students had their own easily communicable way of involving themselves with this model at the very least drew them into the task.

Breaking the task down

As a result of the observations of the first three pairs, the task was simplified by breaking it down into smaller steps, so that after seeing the whole figure in action, the students' task was first to produce two legs of equal length, then two arms and only later bring together all the body-parts in one figure.

- Problematic aspects

  Engagement: The students quickly tired of the activities and after each construction called on the researcher to verify that they had produced what was required. The sense of a challenge to be resolved had disappeared.

Attending to reflection

Observations of students' interactions with the DEG tools up to this point had indicated that the idea of construction was not quickly assimilated. So, given the limited duration of the study, it was not realistic to delay the introduction of tasks specifically related to reflection until after students were completely comfortable with the difference between drawing and constructing. The next introductory task to be trialled involved producing the image of the stick-person under reflection in a given axis (see Figure 5.12).

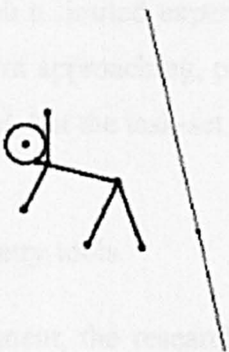

Figure 5.12: The stick-person and a line to be used as an axis for reflection

This time as well as demonstrating the five tools for congruency, students were also introduced the tool `symmetrical point`. The student-pairs who tackled this task made use only of this tool, applying it to each point of the figure and then connecting the points by creating line segments between them or, in the case of the head, using the `circle` tool from the creation menu (and not the `compass` tool). This strategy resulted in dynamic figures such as those in Figure 5.13, which students enjoyed dragging in different ways. One observation of students' dragging behaviour was that they never spontaneously dragged the axis and did so only after interventions from the researcher.

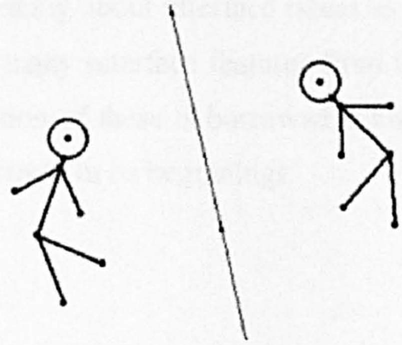

Figure 5.13: The stick person and its image under reflection

For the next task, it was suggested to the pairs that they build their own ways of applying the reflection transformation without using the `symmetrical point` tool. They were reminded how to use the five construction tools for congruency and also shown the `perpendicular line` tool. None of the students was able to complete this activity and it was also extremely difficult to know how to intervene in ways that would help them, without imposing a particular construction method and essentially solving the problem. It was clear that the jump from using to constructing tools was too big for students with such a limited experience of dynamic geometry. With the end of the school summer term approaching, progress had been made on the tool-set of the DEG microworld kernel, but the task-set was still in need of revision.

5.1.2.2 Multiple Turtle Geometry tools

Alongside the DEG development, the researcher also worked with students in two different classes on the multiple-turtle geometry microworld (MTG). Just as a computational environment that has as its model traditional Euclidean geometry might be questioned as a suitable place from which to encourage interfigural considerations, turtle geometry with its emphasis on intrinsic properties might also seem an odd selection. However, with the availability of multiple turtles who can communicate about their relative distances and headings, interfigural relationships can be highlighted and manipulated. Additionally, they offer a means of mathematising the 'plane' by imagining it as consisting of an infinite number of turtles in an infinite number of states (see, Leron and Zazkis 1992; p.325[6]).

Because, unlike Cabri-géomètre, Logo is a programming language, the design of the MTG kernel involved thinking about interface issues as well as constructing a usable tool-set. MTG inherited many interface features from its predecessor Turtle Mirrors (TM) and a brief description of these is borrowed from Hoyles and Healy (1997) to give an idea of the microworld in its beginnings.

6 Leron and Zazkis define turtle states in terms of a triple (x, y, h) where (x, y) are its co-ordinates in a Cartesian system and h its heading clockwise from the north. This study focuses on a conceptualisation in which turtle states are defined in terms of their relationships to other states and not in terms of an underlying Cartesian system.

One important feature of MPB is the existence of *multiple turtles* which can be used in a variety of ways...TM used one blue turtle to draw the starting figure and one red one to draw its image. These two turtles had the same initial position on the mirror line, facing along it (in Figure 1, the blue turtle is hidden beneath the red).

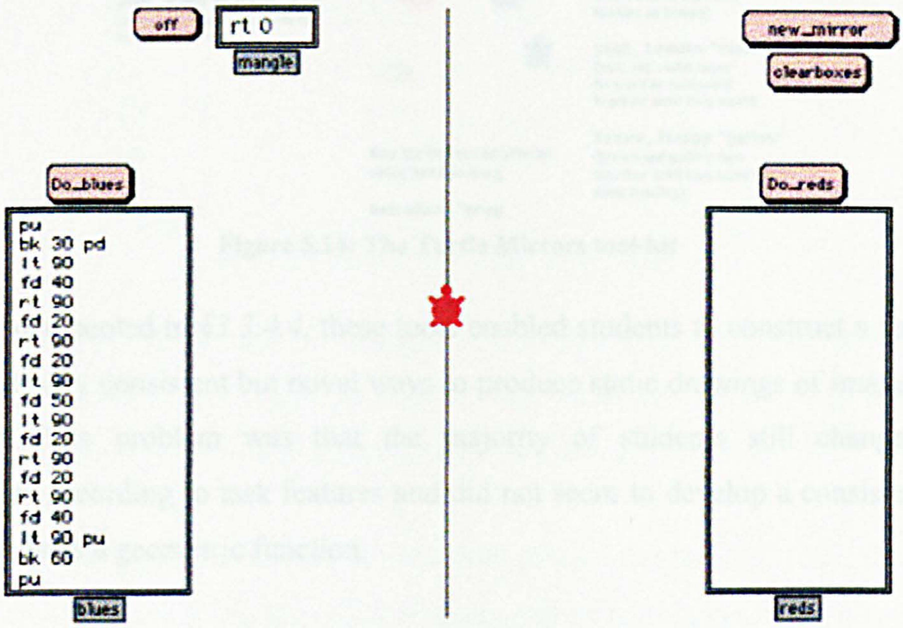

Figure 1. *Reflection in a vertical mirror*

Turning to the screen artefacts of MPB that we used in TM, there are *textboxes*, (the rectangles with frames in Figure 1), which are exploited to promote interaction with the microworld itself and between the students. Textboxes can show information about the specifications of the task that are manipulable; for example, there is a textbox, `mangle`, which shows the current mirror orientation and whose contents can be changed to turn the mirror simply by typing another angle, e.g. `lt 100`. Textboxes can also be used to display a symbolic history of the actions performed in any construction, so that students can focus simultaneously on the symbolic and visual expression of their action...

Finally, the *buttons* on the screen can be pressed to run a command or a procedure and so facilitate the easy management of the work space.

(Hoyles and Healy, 1997; p.33)

In addition to the various screen artefacts of TM, a number of Logo commands were constructed, modified or simply highlighted in a tool-kit presented to students (although students were also free to use any other Logo commands they chose). The original TM tool-kit is shown in Figure 5.14.

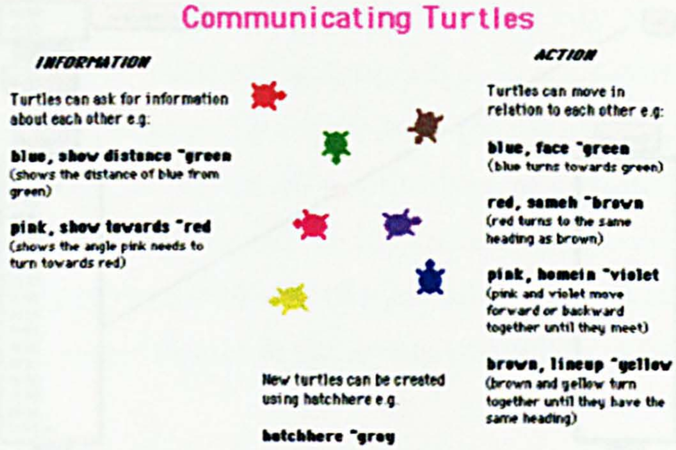

Figure 5.14: The Turtle Mirrors tool-kit

As was documented in *§3.3.4.4*, these tools enabled students to construct a variety of mathematically consistent but novel ways to produce static drawings of images under reflection. The problem was that the majority of students still changed their approaches according to task features and did not seem to develop a consistent view of reflection as a geometric function.

The first MTG kernel

In this ground-laying iteration, a number of modifications were made to transform TM into MTG. The major concern was to support moves from seeing reflection in a series of particular visual designs to focussing on a sense of generality.

The first modification was relatively simple. Generality ought to imply generality to all turtle states, including those used to define the axis of reflection. Since turtles, unlike Euclidean points, have headings as well as positions, they can be thought of simultaneously as points and as lines. In MTG, just as in its predecessor, a visual trace of the axis was produced by one of turtles (named mirror), but in other senses this turtle was the same as any other. For this reason, the `mangle` textbox from TM was replaced by the textbox `mirrors` and the button `Do_mirrors`. These screen objects could be used in combination to record and run a list of commands for the mirror turtle (see Figure 5.15).

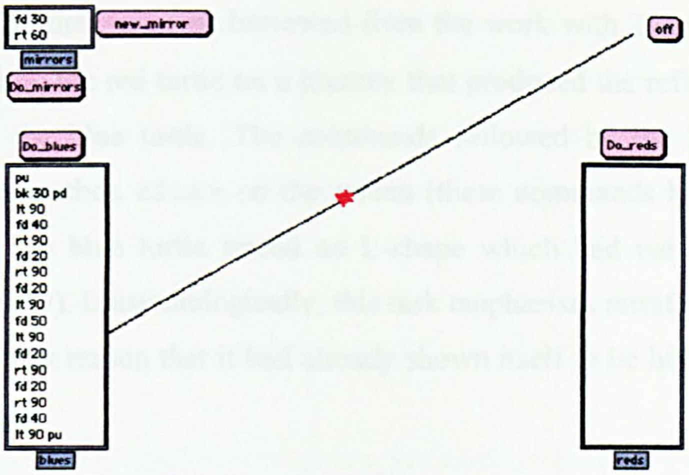

Figure 5.15: Screen with three turtle textboxes

A second change was to add a command `flip` following that used by Leron and Zazkis (1992) which turned a turtle onto its back, with the effect that to subsequent commands to go right the turtle would turn to the left and vice versa.

A third modification to the TM tools involved two of the turning tools, `face` and `sameh`, both of which had the effect of executing a turn without its size necessarily made explicit. This had meant that access to its value was not later available to repeat the turn. With this in mind, the `face` tool was deleted and the `sameh` tool was modified to become `toheading`, a tool for information, outputting the turn necessary for one turtle to have the same heading as a second without actually executing the command.[7]

The MTG tool-kit available to the students at this stage is presented in Figure 5.16. Before attempting the first task, students were shown how to access the page in which the tools were displayed. They were also introduced to various screen features of the microworld and reminded about the tools `rt`, `lt`, `fd`, `bk` and `cg` (clear-

[7] A number of interface modifications were also attempted, involving placing the tool-set as icons on the screen that could be executed directly as well as under program control. The majority of students responded negatively to this changes and indicated that they preferred typing in the commands. Hence, the interface modifications were eventually rejected.

graphics)[8]. The first task was borrowed from the work with TM and involved the students in sending the red turtle on a journey that produced the reflected image of a path drawn by the blue turtle. The commands followed by the blue turtle were accessible in the textbox blues on the screen (these commands had been changed slightly so that the blue turtle traced an L-shape which had neither rotational or reflective symmetry). Epistemologically, this task emphasises intrafigural aspects and perhaps it is for this reason that it had already shown itself to be highly accessible to students of this age.

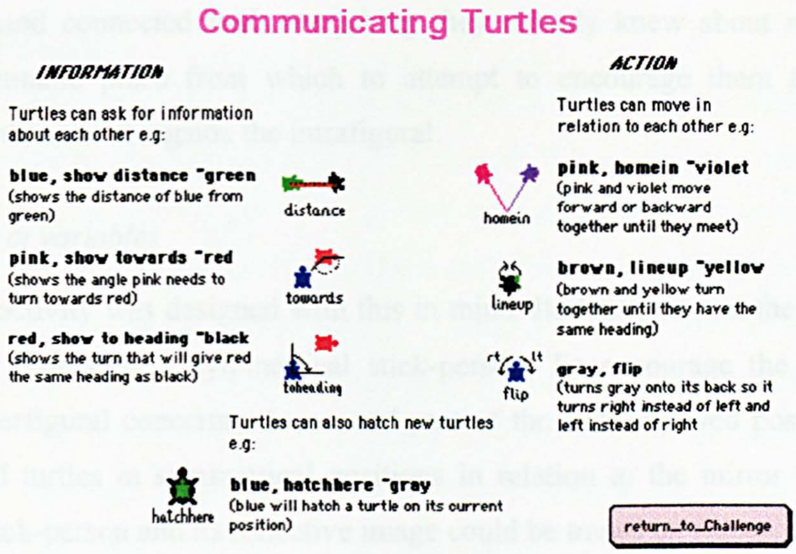

Figure 5.16: MTG tool-kit (first kernel)

- Favourable aspects

  Tools: Students quickly appropriated the interface tools as they worked on the activity. All student-pairs adopted the method of swopping the direction of the turns, so that left turns become right ones and vice versa. The description of one of the pairs is presented in Figure 5.17 below.

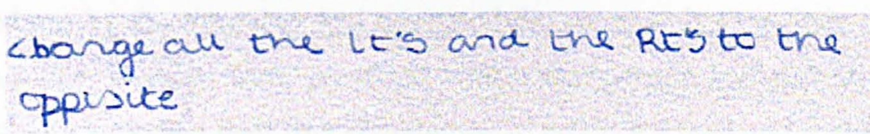

Figure 5.17: Describing the process of constructing an image under reflection of a Logo figure

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[8] All the students had used Logo before, but their experience had been basically limited to the construction of pictures using these four commands.

The success of the task was that all students were able to use the tools of the microworld to express a mathematical relationship, an abstraction (grounded in the context of generation and tools for mediation and formalisable in terms of the Logo language). However, it lacked generality in that it only applies when the task has certain features. To be put into practice, it requires two turtles initially positioned in the same state on the axis of reflection and it requires the presence of a symbolic description of the trajectory of one of the turtles.

Nonetheless, since this particular abstraction appeared to make a great deal of sense to students and connected with everything they already knew about reflection, it seemed a suitable place from which to attempt to encourage them to focus on interfigural relations alongside the intrafigural.

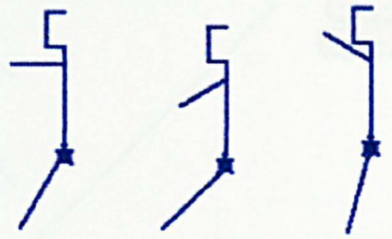
Introduction of variables

The second activity was designed with this in mind. Its first part had the intrafigural objective of producing a symmetrical stick-person. To encourage the students to attend to interfigural concerns, the second part of the task involved positioning the blue and red turtles in symmetrical positions in relation to the mirror turtle, from which the stick-person and its reflective image could be traced on screen.

- Problematic aspects

  Engagement: Students became very caught up in the details of the first half of their person. This was not necessarily a problem for them: they seemed motivated to continue despite experiencing some difficulties in recording correctly the commands that produced the trace they were hoping for. But this could take a considerable length of time, so students had often forgotten about how to write procedures by the time they had generated the commands to include in one. To resolve this, students were presented with a ready-made half-person procedure (see Figure 5.18) and their first activity was to produce a method for obtaining the second half.

```
to per :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```



```
:sh  = 90     :sh  = 60     :sh  = 120
:hip = 30     :hip = 60     :hip =  15
```

Figure 5.18: The procedure for the Logo stick-person

- Favourable aspects

Tools: An advantage of presenting students with a half-person procedure was that it introduced a procedure with variable inputs. When students were writing a procedure for the second half of the person, they treated the turns associated with variables in exactly the same way as those with fixed values. Perhaps because this method could be applied mechanically, or perhaps because the idea of swopping turns is highly intuitive, students rarely even stopped to comment on the variables as they carried out this change. It also seemed to be entirely obvious to them that the same variable values should be assigned to corresponding procedures when they were executed.

Engagement: Although not related to mathematical aspects of the tasks, it was noted that the naming of procedures often provided a light-hearted moment at which students could personalise their own computer constructions. All students followed the researcher's lead in giving two halves of one name to the respective half-person procedure (`bob` and `by`, for example). It also seemed important to students to preserve symmetry when constructing symmetrical people: they invariably swopped the order of the half-person procedures which produced the image (see commands in `blues` and `reds` textboxes in Figure 5.19).

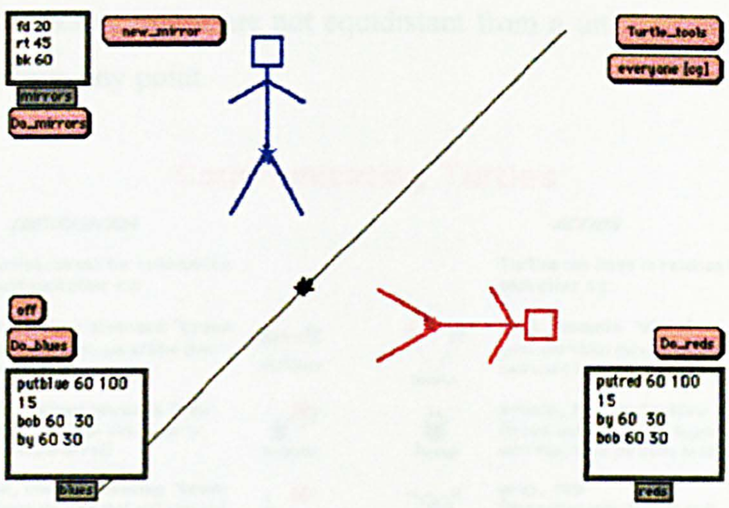


Figure 5.19: A person and its image

So, the positive aspect of the stick-person task was that it had enabled students to include the use of variables (as generalised numbers) into their activity. On the slightly concerning side, students had as yet used none of the tools for communicating between two turtles and it was becoming evident that the tool `flip` was not proving as useful as had been expected. The idea of turning the turtle belly-up did not seem to be particularly appealing. Only one student had chosen to use it and even she had problem: after flipping the blue turtle, she later forgot it had been turned upside down and had difficulties in interpreting its behaviour. Perhaps because it was not visually obvious that there was any difference between a flipped or unflipped turtle, students felt in more control if they left the turtle in the original position state – deciding which way is left and which is right isn't immediately obvious to everybody, so also to have to reverse this mentally seemed to be too much effort. Some students did say that they would like was a tool which swopped the turns in any command list given to a turtle without altering the turtle's state. This provided the inspiration for a new tool `swop`.

A task for communicating turtles

In search of a task that would encourage use of the now modified communication tools (Figure 5.20), an activity was devised in which, having produced a figure including a person and its image, students would be asked to find different ways of reuniting the red and blue turtles at a coincident turtle state on the axis. In this activity students could experience the general relationship that a point (or, in this case, turtle state) and its reflective image are not equidistant from a unique point on the axis or reflection, but from any point.

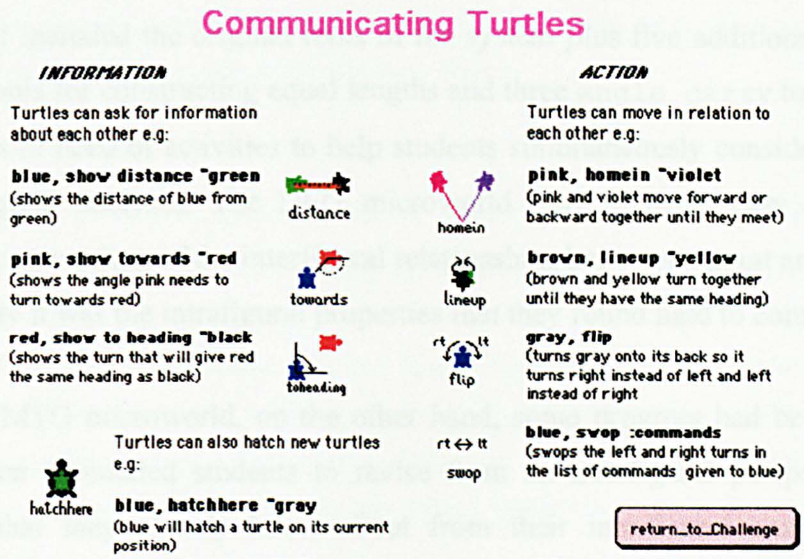


Figure 5.20: MTG toolkit at the end of first iteration

- Favourable aspects

  Engagement: The notion of reuniting turtles seemed to be an attractive one to the girls. Most pairs produced more than one method to do this, with the maximum number four. These methods involved different combinations of the tools, `homein`, `distance`, `towards` and `toheading` (not all were used in each method). The `homein` tool, which had been built to model an intuitive notion of an interfigural nature – that when symmetrical turtles move towards each other they meet on the axis of reflection – was frequently used, with some students quite enchanted by the sight of the two turtles "*scuttling*", "*running*" or "*crawling*" towards each other. Some pairs used the strategy of sending both turtles back through the commands that had positioned them. Although this strategy did not highlight any new properties, this was not an issue since students saw it as only one of a number of ways of using equal angles and distances to reunite the turtles.

Despite the success of this task, after it had been completed the iteration ground to a halt. The tools had been successfully used in the construction and validation of specific images, but were still insufficient to enable the formalising of a reflection function.

5.1.2.3 The DEG and MTG microworlds at the end of the first iteration

By the end of the ground-laying iteration, the tool-set of the DEG microworld was complete (it included the original tools of the system plus five additional tools, two `compass` tools for constructing equal lengths and three `angle carry` tools), but the task-set was in need of activities to help students simultaneously consider both intra and interfigural analyses. The DEG microworld gave students easy access to a dynamic representation of the interfigural relationships between a point and its image, but ironically it was the intrafigural properties that they found hard to construct.

Within the MTG microworld, on the other hand, some progress had been made on activities that stimulated students to revise from an interfigural perspective those properties that they already knew about from their intrafigural analyses. Some additions to the tool-set, however, were still needed. The existing set allowed students to write a general procedure if the pre-image turtle was initially located on top of the mirror turtle, but would not support students in formalising a procedure for the more general case.

5.1.2.4 Learning processes in the microworld

The focus in microworld development during this iteration was primarily on the provision of suitable tasks and tools, but, as these were tested, some observations about important aspects of the learning process were made. In particular, it was noted that in the course of their interactions, students were coming up with generalised descriptions that have been defined as *situated abstractions* by Noss and Hoyles (1992; 1996; also, Hoyles & Noss, 1992). The description shown above in Figure 5.17 provides a good example. Situated abstractions involve a reflection (in the thinking sense) by learners on the operations used to solve a set of problems and as such are similar to reflective abstractions as described in *§2.1.1*. The difference is that while reflective abstractions have traditionally been associated with a process of decontextualisation, the use of the qualifier *situated* is intended to emphasise that abstracting activities are related to, and articulated in terms of, the medium in which they are constructed (Hoyles and Noss, 1993; p. 84). Rather than decontextualisation, they describe the process of situated abstraction as

> "...a process of connection to new objects, a process that develops in activity: abstracting within a domain rather than away from it"
>
> (Noss and Hoyles, 1996; p.226)

Lest this seems to emphasise a process of concretion rather than abstraction, it is important to stress too their view of situation abstractions as objects that are

> "... abstractions in two senses. First they operate beyond the specific experiences in which they arise (compare with Vergnaud's (1982) delimitation of "theorem" to the realm of action or Lave's (1988) focus on practice)...Second we imply that there is some conscious appreciation of the generalised relationships among the concepts involved.
>
> (Hoyles and Noss, 1992; p.43)

Another way of describing this is to say that situated abstractions are about identifying relationships, concrete in the particular domain in which they are made *and* which might be concretisable in other domains as well.

The second issue emerging from observation of student microworld activity went beyond entirely cognitive aspects. It was clear that some aspects of the tools and tasks were associated with different affective reactions (ranging from displays of pleasure, to anxiety and irritation). In particular, the most positive reactions to the microworld activity seemed to occur when students were able to juxtapose formal expression with experienced activity. For example, the dancing person whose hips described a circle helped students give meaning to the `compass` command, while the functioning of the `symmetrical point` tool remained relatively opaque. As well as suggesting a design principle to be considered in the tool and task development of the next iteration, this observation also suggested another focus for the analysis of learning systems in Phase 2: a consideration of the features of tools and tasks that helped the students to *want* to engage with them.

5.2 Iteration 2: Piloting of tests, tools and tasks

The activities of the first iteration highlighted a number of issues regarding student knowledge of reflection suggesting that, in paper-and-pencil activities, students engage in little mathematical analysis of an interfigural nature and focus their attention instead on the relationships between internal properties of the pre-image and image. One question for the second iteration was whether the interpretations about students' views of reflection gleaned from a limited number of interviews applied to a larger population, and whether similar descriptions and constructions would be obtained if students completed a written paper-and-pencil test.

The analysis of the preliminary versions of the microworlds in the first iteration suggested that the tools of each kernel offered students new ways to consider intrafigural and interfigural relationships, different both from each other and from the paper-and-pencil context. Relatively little attention had been focussed on controlling the tasks so that the mathematical demands were reasonably equivalent. And tasks demanding a formalisation resembling a reflection function had yet to be incorporated into the microworld activities. These issues were addressed during this second iteration.

5.2.1 The pilot paper-and-pencil test

In their pilot state, two versions of the paper-and-pencil test were devised. Both versions included the same initial question, asking students to write everything they knew about reflections and reflective symmetry. The tests also included a set of six construction items, asking students to sketch images of various figures under reflection (see Figure 5.21). Some changes were made to the items used in the paper-and-pencil test as compared to the interview items. First, the pre-images included figures more complex than segments. Second, in the case of the figure crossing the axis of reflection, it was conjectured that the need to 'break' the segments into two parts might play a part in encouraging its translation rather than reflection and a two segment figure was used instead. Additionally, two figures were included whose vertices did not lie on the grid-points to consider further the mediating tool of this reference system.

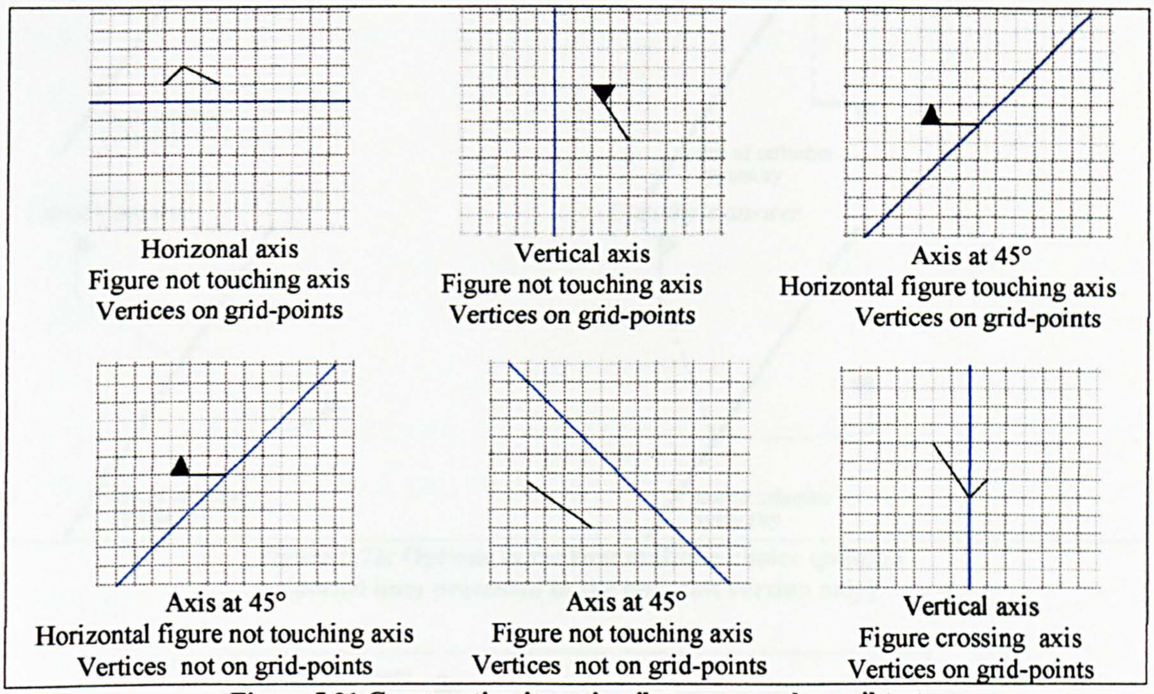


Figure 5.21 Construction items in pilot paper-and-pencil test

The only difference between the two versions of the test related to the final part. This involved two multiple-choice questions in which students were asked to select one from a variety of possible representations of a pre-image and its image under reflection. In one version of the test (henceforth *pink test* since it was printed on pink paper), the options were presented with dotted construction lines attempting to show the process used to locate the image (Figure 5.22 and Figure 5.23). In the second version of the test (the *blue test*), the items were the same except that the dotted construction lines were not shown. The rationale was that the dotted lines provided an explicit sign of the interfigural relationships between pre-image, axis and image. Hence their presence was appropriate in a filling-inwards version of the test but not for the filling-outwards one.

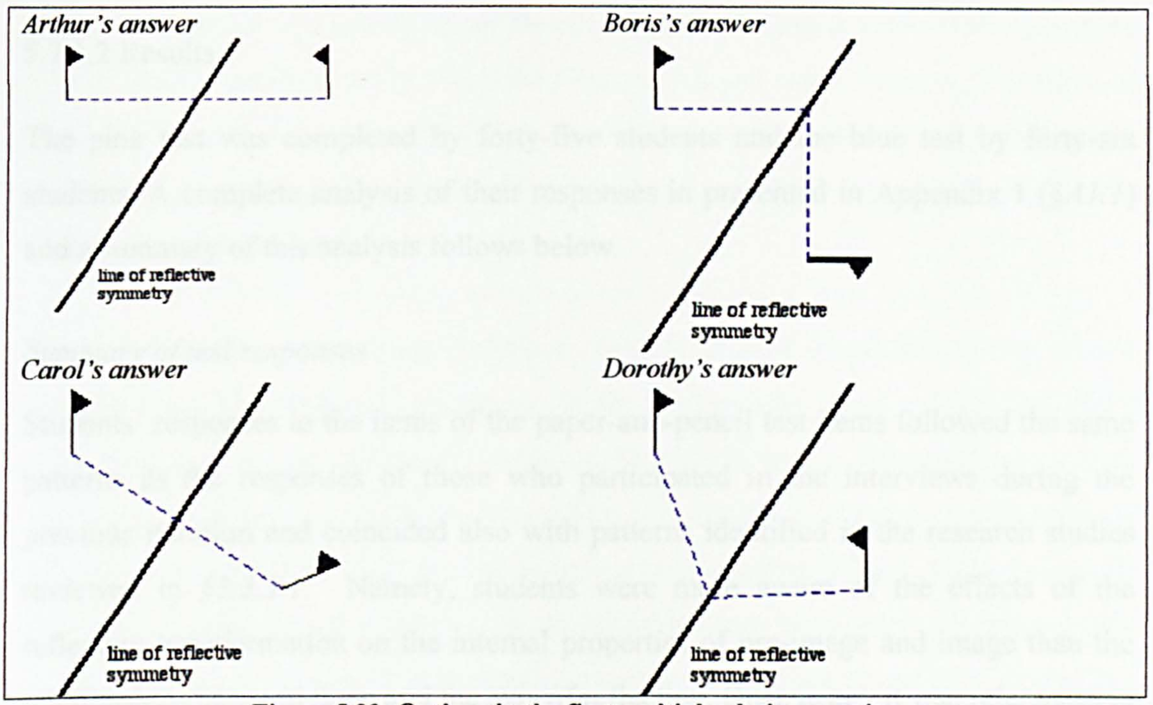


Figure 5.22: Options in the first multiple-choice question (dotted lines presented in the pink test version only)

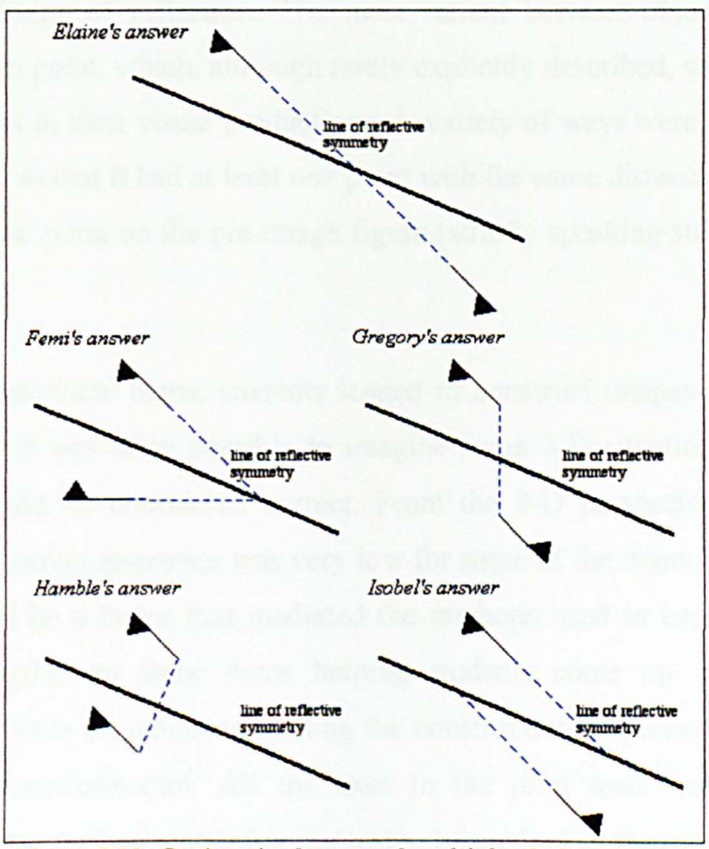


Figure 5.23: Options in the second multiple-choice question (dotted lines presented in the pink test version only)

5.2.1.2 Results

The pink test was completed by forty-five students and the blue test by forty-six students. A complete analysis of their responses in presented in Appendix 1 (§*A1.1*) and a summary of this analysis follows below.

Summary of test responses

Students' responses to the items of the paper-and-pencil test items followed the same patterns as the responses of those who participated in the interviews during the previous iteration and coincided also with patterns identified in the research studies reviewed in *§3.3.3.1*. Namely, students were more aware of the effects of the reflection transformation on the internal properties of pre-image and image than the relationships between them and the axis of reflection. Once again, it was references to intrafigural relationships that characterised students' written descriptions of their personal knowledge of reflection. The most salient between-object property was distance between point, which, although rarely explicitly described, students did try to take into account in their visual productions. A variety of ways were used to position an image-figure so that it had at least *one* point with the same distance from *one* point of the axis as *one* point on the pre-image figure (strictly speaking still an intrafigural treatment).

Just as in the interview items, students tended to construct images as congruent to pre-images and it was often possible to imagine some 3-D situation in which their construction could be considered correct. From the 2-D perspective however, the percentages of correct responses was very low for some of the items. The presence of the grid seemed be a factor that mediated the methods used to construct particular angles and lengths, on some items helping students come up with the correct representation, while on others suggesting the construction of properties unrelated to the reflection transformation. All the axes in the pilot tests were orientated at multiples of 45° to the horizontal. For the test to be applied in Phase 2, it was decided to include an item in which the axis was orientated at a different angle, with the conjecture that the mediational effects of the grid would result in very few correct responses.

The test responses also indicated that the majority of students acted as if the reflection transformation applies to only one side of the axis (in the case of item 6, the left-hand side). For the Phase 2 version of the paper-and-pencil test, this item was modified, to avoid the 'almost' overlap of the segments under reflection, which could have accentuated this tendency.

Finally, analysis of students' choices suggested that providing some visual representation of interfigural properties might have had an effect on their decisions about reflective images, although it was not possible to assess from the responses to the multiple-choice questions the extent to which students saw these properties as general. The difference between the tests had been motivated by a desire to present the students with general properties in the filling-inwards but not the filling-outwards version. However, like any paper-and-pencil drawing, whether or not the construction lines were seen as general depended not on the drawing itself but on the students' relationship with it. In retrospect, it was felt that simply providing an external sign of a relationship was not sufficient to count as a filling-inwards intervention. It could equally be considered as providing empirical data in a filling-outwards approach. This suggests a problem in creating tasks specific to one or other instructional approach and it was decided that the differences in instructional approach would be limited to the teacher interventions before, during and after the tasks. On the basis of these observations, only one version of the paper-and-pencil test was prepared for Phase 2, in which the multiple-choice questions would include construction lines, illustrating relationships between pre-images, axes and images, but also with equal angles and distances marked where appropriate.

5.2.2 The computer-based tasks

As well as the continued efforts to probe students' knowledge about reflection in paper-and-pencil contexts, this iteration also involved more work on task and tool design. There were basically two objectives behind the research activities related to this aspect. First was the need to design a task specifically structured to move students' focus between intra and interfigural aspects, a need reinforced by the test results. Second was attention to task sequencing and the structuring of the activities to ensure as close a correspondence as possible (in terms of mathematical demands) between tasks of the DEG and MTG learning systems.

5.2.2.1 Encouraging movement between intra- and interfigural analyses

The task developed with the aim of supporting movement between intra and interfigural concerns was modified from one applied in a Logo context by Graf (1988) and later adapted for use with Cabri by Healy, Hölzl, Hoyles and Noss. In its previous form, the task had involved discovering the types of quadrilaterals that could be produced when a triangle is reflected in one of its three sides (an intrafigural activity). For this study, the task had a second part in which students were to use the properties of the quadrilateral to define their own methods of constructing images of points or turtles under reflection. In this way, it was hoped that any intrafigural relationships would be revisioned from an interfigural perspective.

The DEG version

For the DEG learning systems, the task involved the creation of a general triangle ABC which would form the basis of a quadrilateral $ABCA'$, where A' was the reflection of A in $\overleftrightarrow{BC}$ (see Figure 5.24). By dragging the vertices A, B or C, some quadrilaterals could be obtained, while others were impossible. Students were asked to try to produce various well-known quadrilaterals, with the eventual intrafigural goal being the identification of the properties all possible quadrilaterals have in common.

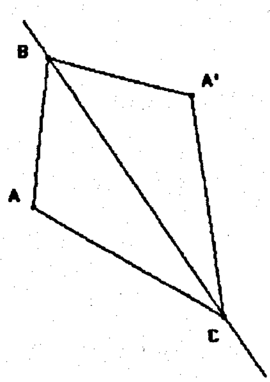


Figure 5.24: A quadrilateral formed by reflecting one vertex of a triangle in its opposite side

In the second part of the task, students were asked to find different ways of constructing a point equivalent to A' without making use of the `symmetrical point` tool of the DEG kernel.

As this task was trialled, attention was given to the kinds of interventions that seemed to help students in activities related to the intended mathematical knowledge and, especially, to draw their attention to the use of construction and validation tools. These interventions included the following:

- Encouraging students to drag all the free points of their construction and not just point A.
- Introducing the use of the locus tool to obtain a trace of the constructed point A' as the free points were moved.
- Suggesting that students make use of DEG tools to aid in attempts to create particular properties by eye. For example, if students were struggling to drag the A so that AC was parallel to BA', they could be shown how to construct a line parallel to the required segment and onto which A could be placed (see Figure 5.25).

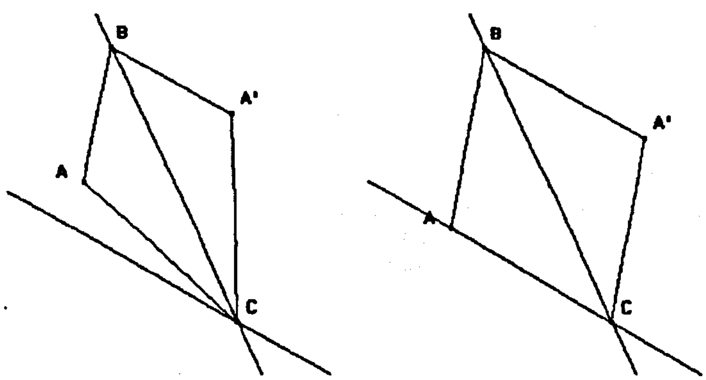


Figure 5.25: Using a construction to guide dragging

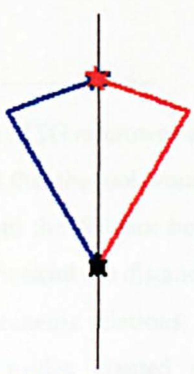
- It would also have been possible to introduce students to the check-property tools and to the measuring tools, but it was decided that, where students wanted to verify properties that they 'saw' on screen, the first intervention would be to suggest they check using a construction tool (for example, a `compass` tool could be used to check whether segments were the same length). In practice, many students actually asked about measuring and when they did they were shown the relevant tools.

During the second part of the task, observations of students' interactions indicated that, despite building up a sense of exactly where A' should be located, not all students chose to position it by means of the construction tools. Some preferred to use the measuring tools of the DEG system to drag the point very carefully into place rather than to build robust constructions that did not mess-up when dragged. These preliminary observations suggested that students were incorporating both intra and interfigural analyses in their work, but were choosing to draw geometrical properties visually rather than produce figures with an underlying DEG 'text'. The apparent emphasising of action and its perceptual effects over symbolic expression in students' DEG interactions was hence identified for close attention in the comparison stage.

The MTG version

For students using the MTG kernel, the construction of a triangle with two vertices on the axis of reflection was in itself quite a challenge. One strategy was to use the `homein` tool and draw both the initial triangle and its image simultaneously as shown in Figure 5.26.

Figure on screen	**Logo code**



```
blue, lt 30 fd 100 rt 100
red, rt 30 fd 100 lt 100
blue, homein "red
```

Figure 5.26: A Logo quadrilateral made up of two symmetrical triangles

Another method was to draw one side of the triangle and then hatch a second turtle, which could be inched forward until it appeared to meet the axis (Figure 5.27).

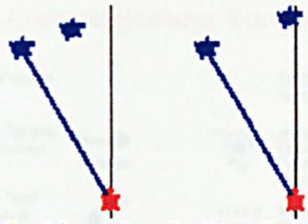


Figure 5.27: Inching a hatched turtle towards the axis

Both these methods involved locating turtles at the point at which their paths met – a general version of the `homein` tool, in fact, and a multiple-turtle version of the point of intersection of two lines. A new tool `meet` was developed, which, in its final version, simulated the action of turtles moving successively closer[9]. Figure 5.28 shows various snapshots of the `meet` tool in action. One other design decision was also made: to include in the tool the action of hatching a new turtle that would mark the meeting point[10].

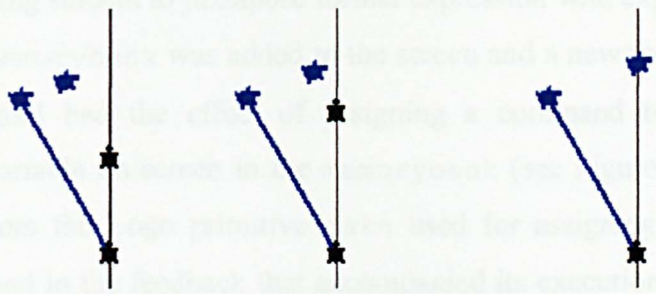


Figure 5.28: Three snapshots of the `meet` tool in action (`blue, meet "mirror`)

The `meet` tool finalised the communicating tool-kit (Figure 5.29). Its inclusion was accompanied by the removal of the `homein` and `lineup` tools.

9 In the spirit of the MTG microworld and its geometry based on relative positions between turtles, it had been intended that the tool would involve two turtles moving forwards or backwards iteratively closer together until the distance between them was negligible. In practice, the resulting procedure was too slow and instead the distances of the two turtles from their meeting point was calculated on the basis of trigonometric relations.

10 The naming of turtles created under programming-control is an issue in multiple turtle environments. For the microworld, the convention adopted was that new turtles would inherit the same colour as the turtle active at the moment of their birth, and they would be named by adding the lowest number not previously used to the colour. So, if blue was active, blue1 would be the first 'child' of the blue set, subsequent blue turtles would be named blue2, blue3 etc.

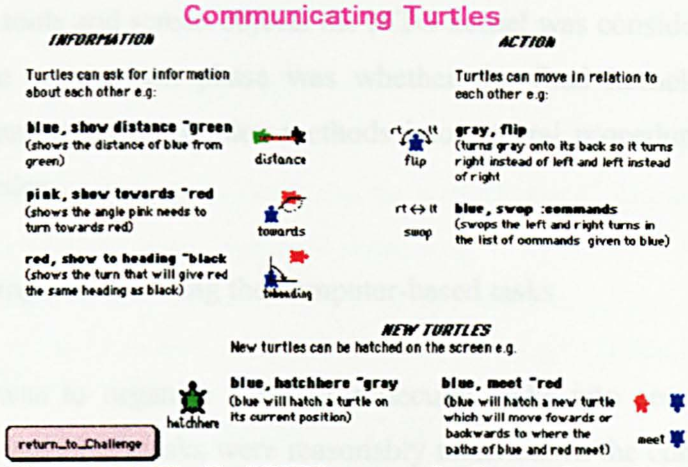


Figure 5.29: The final MTG communicating turtles tool-kit

However, the microworld kernel was still not entirely complete. The next step was to provide an accessible way for students to record and recall the information returned by the communication tools. Following the design principle identified in the previous iteration of enabling student to juxtapose formal expression with experienced activity, a new text-box `memorybank` was added to the screen and a new tool `remember` was designed. This tool had the effect of assigning a command to a variable and displaying this variable on screen in the `memorybank` (see Figure 5.30). It was not very different from the Logo primitive `make` used for assigning global variables, except in name and in the feedback that accompanied its execution. Using this new tool, it was now not only possible for students to concretise the process of assigning variables through connecting it with acts of remembering, but they could also see the value of any variables as they were remembered.

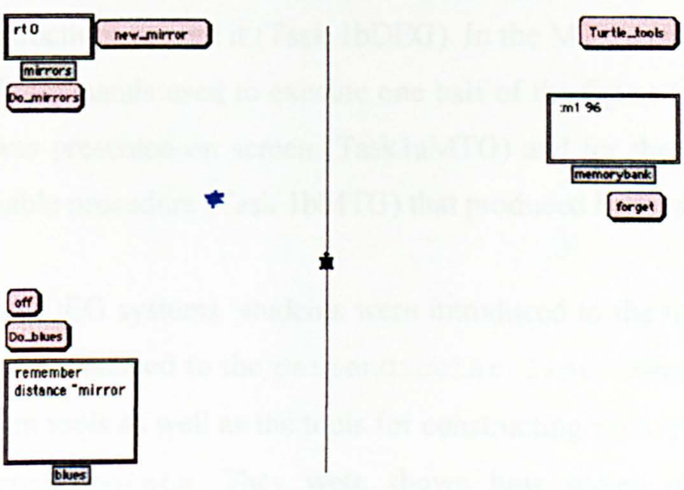


Figure 5.30: Remembering a distance between turtles

With these new tools and screen objects the MTG kernel was considered complete. A question for the comparison phase was whether the final kernel was usable by students to formalise quite complex methods into general procedures that could be applied to all turtles.

5.2.2.2 Sequencing and matching the computer-based tasks

The final step was to organise a set of reflection tasks into sequences in which respective DEG and MTG tasks were reasonably matched. In the conception of these sequences, various forms of different tasks were trialled by a total of thirty-three students who spent from 1hr and 30 minutes to 6 hrs on the microworld activities (fourteen tested the DEG tasks and tools and seventeen worked with the MTG versions). The final sequence comprised of five tasks, which are presented below.

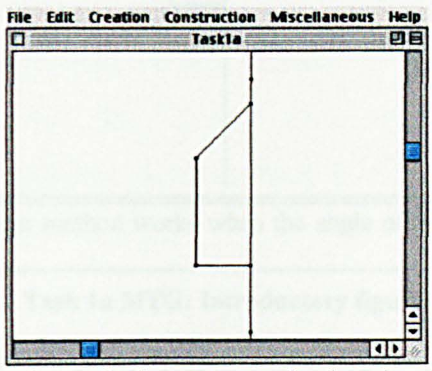
Task 1: Symmetrical figures

Computer activities: The first task involved students in completing figures given one half and the axis of reflection. Before working on two figures, the first a geometric shape and the second a stick-person, students were introduced to a subset of the microworld tools and given a little time to experiment with aspects of its interface.

Both DEG and MTG versions had two parts. In the DEG systems, students had to complete a figure given just its visual representation (Task 1aDEG) and were then presented with an incomplete stick-person figure along with a symbolic record showing the constructions behind it (Task 1bDEG). In the MTG systems, for the first figure, the list of commands used to execute one half of the figure (i.e., its symbolic representation) was presented on screen (Task1aMTG) and for the second, students were given a variable procedure (Task 1bMTG) that produced half a stick-person.

New tools: For the DEG systems, students were introduced to the tools of the Cabri creation menu and introduced to the `perpendicular line`, `compass` and `angle carry` construction tools as well as the tools for constructing `points on objects` and `intersection points`. They were shown how screen objects could be manipulated through dragging. In the second part of the task the `exposition` tool was demonstrated. For the MTG systems, the turtle-moving tools (`fd`, `bk`, `rt`, `lt`, `pu`, `pd` and `cg`) and the tools in the communication tool-kit were presented to students, with particular emphasis on `swop` and `flip`. Use of the text-boxes and buttons of the screen interface was also demonstrated. For the second part of the task, students were also introduced to the writing of variable procedures.

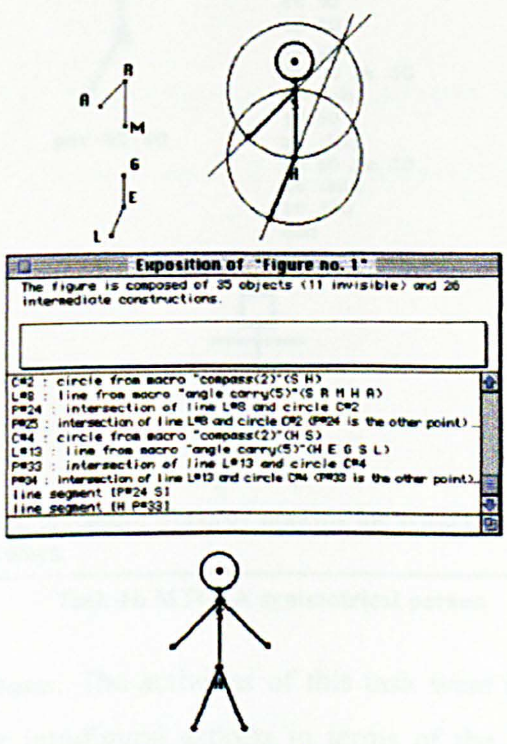
Complete the design on your computer screen by constructing the reflected shape.



Check that your image is always correct by dragging the line.

Task 1a DEG: Introductory figure

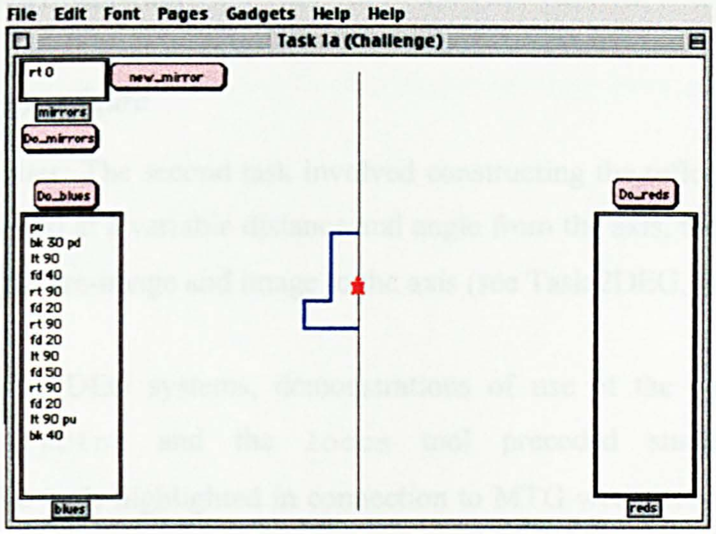
Use the Cabri description of half a person to help in the construction of a complete symmetrical stick-person.



Check the figure remains symmetrical even when it is dragged.

Task 1b DEG: A symmetrical person

Complete the design on your computer screen using the red turtle to trace the reflected shape.



Check that your method works when the angle of the mirror turtle is changed.

Task 1a MTG: Introductory figure

Use the Logo procedure for half a person to help in the construction of a complete symmetrical stick-person.


```
to per :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

Check your procedure works by drawing the symmetrical person in different ways.

Task 1b MTG: A symmetrical person

Epistemological emphasis: The activities of this task were intended to focus on the expression of familiar intrafigural aspects in terms of the tools of the microworld kernels. The intended mathematical knowledge associated with the task concerned the within-figure properties conserved under reflection (congruency of pre-image and image, reversal in orientation).

Task 2: Reflecting a figure

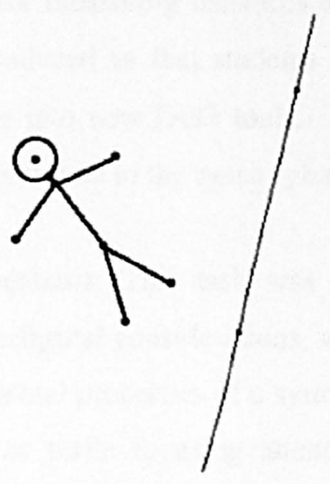
Computer activities: The second task involved constructing the reflective image of a stick-person located at a variable distance and angle from the axis, then exploring the relationship of the pre-image and image to the axis (see Task 2DEG, Task 2MTG).

New tools: In the DEG systems, demonstrations of use of the construction tool `symmetrical point` and the `locus` tool preceded students' computer explorations. The tools highlighted in connection to MTG were `hatchhere`, `meet`, `distance`, `towards` and `toheading`.

Epistemological emphasis: This task was intended to focus attention on interfigural relationships between figures. For both tasks, students had support in the construction of image 'points'. The intended mathematical knowledge associated with the task concerned the between-figure properties conserved under reflection (distance and orientation with respect to the axis).

Construct the image of the stick-person by reflection in the line on your screen.

Check your method works for different positions of the person and the line.

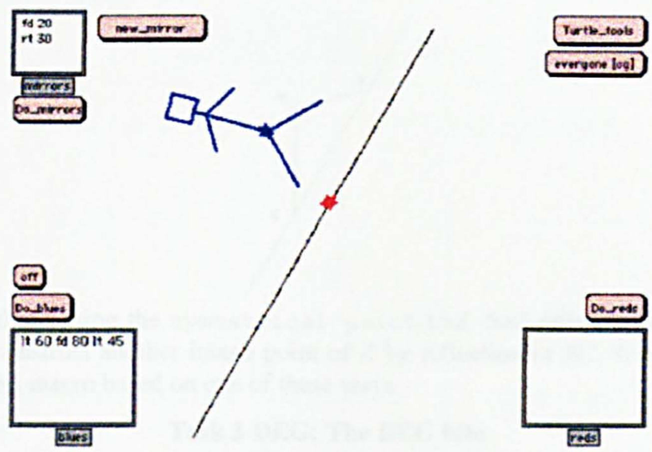


Investigate what happens to points on the image when points on the line or original figure are moved. Can you move a point from the original stick-person so that it is on top of its image?

Task 2 DEG: Reflecting people

Write ONE procedure that can be used to send blue to different distances and angles from the mirror line. Use blue to produce the stick-person and red to construct its image by reflection in the mirror line.

Check your method works for different positions of the person and the mirror.



Find different ways to reunite red and blue on the mirror line.

Task 2 MTG: Reflecting people

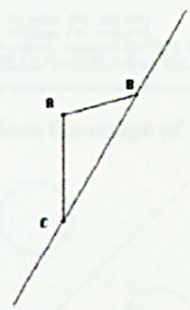
Task 3: From within to between

Computer activities: The bridging task was described extensively in *§5.2.2.1.* Task 3DEG and Task 3MTG below illustrate its final presentation to the students.

New tools: In the DEG systems, the tools used depended on the particular activities of the students, but could include new construction items (for example, `parallel lines`) and tools for measuring distances and angles. It was also intended that the `macro` tool be introduced so that students could turn a procedure for constructing symmetrical objects into new DEG tools. In the MTG systems, the students were introduced for the first time to the `memorybank` textbox and its related tools.

Epistemological emphasis: This task was intended to act as conduit connecting intrafigural and interfigural considerations, with the investigation of the quadrilateral emphasising the internal properties of a symmetrical figure, and the construction of a symmetrical point or turtle focusing attention on the same properties, but in an interfigural form. The formalisation in microworld tools of their symmetrical point/turtle construction was intended to encourage ideas related to reflection as a geometrical function. In addition, it was hoped that students would begin to see points or turtles as general dynamic objects with variable positions, rather than specific instances.

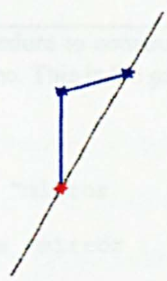
What kinds of quadrilaterals can be made by reflecting one point (A) of a triangle in a line passing through its other two points (B and C)?



Without using the `symmetrical point` tool, find different ways to construct another image point of A by reflection in BC. Write a Cabri macro based on one of these ways.

Task 3 DEG: The DEG kite

What kinds of quadrilaterals can be made by reflecting a triangle which has one side along the mirror line?



Without communicating with the red turtle, find different ways to position a new turtle so it is the image of blue by reflection in the mirror line. Write a Logo procedure based on one of these ways.

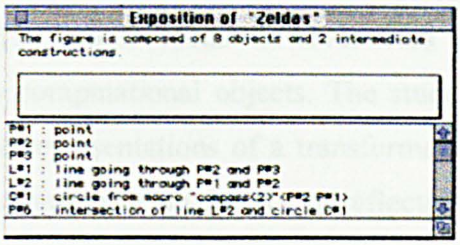
Task 3 MTG: The MTG kite

Task 4: Interpreting computer constructions

Computer activities: In the fourth task, students were presented with the construction of a fictional student, Zelda. The construction was given in both visual and symbolic forms. The students had to decide whether Zelda's construction produced reflections and if not to modify it.

New tools: No new tools were introduced during this task.

Zelda built her own tool to construct the image of a point by reflection in a line. These were the commands she used:

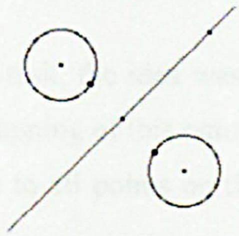


Exposition of "Zeldas"

The figure is composed of 8 objects and 2 intermediate constructions.

P#1 : point
P#2 : point
P#3 : point
L#1 : line going through P#2 and P#3
L#2 : line going through P#1 and P#2
C#1 : circle from macro "compass(2)"(P#2 P#1)
P#6 : intersection of line L#2 and circle C#1

She used her tool to produce the image of a point on a circle.



Check whether Zelda's tool produces the correct image-point always, sometimes or never.

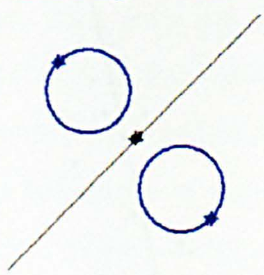
If necessary, write a modified version of the tool for Zelda.

Task 4 DEG: Zelda's tool

Zelda built her own procedure to construct the image of a point by reflection in the mirror line. This is the procedure she wrote:

```
to zeldas
blue, hatchhere
remember towards "mirror
run :m1 rt 180
remember distance "mirror
bk :m2
remember toheading "mirror
run :m3
swop :m3
bk  :m2
swop :m1
end
```

She used this to position an image turtle and trace an image circle.



Check whether Zelda's procedure produces the correct image-turtle always, sometimes or never.

If necessary, write a modified version of the procedure for Zelda.

Task 4 MTG: Zelda's procedure

Epistemological emphasis: This task was intended to illustrate that the reflection transformation is only one of a group of isometries; as such it touched upon transfigural concerns. The major focus however was on the formalisation of interfigural properties as computational objects. The students were to interact with both visual and symbolic representations of a transformation, in fact representing a rotation of 180° around a point/turtle on the axis of reflection.

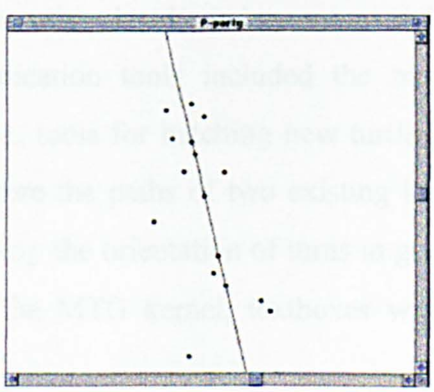
Task 5: Mappings of the plane

Computer activities: In the final task, the idea was that students would focus on the plane and enact reflection as a mapping of this onto itself. The intention was to stress application of the transformation to all points on the plane and not just to particular points or figures.

New tools: No new tools were introduced in the DEG system. In the MTG system, the tool `everyone` (a primitive which allows the user to give the same command to all the turtles on the screen) was highlighted for potential use.

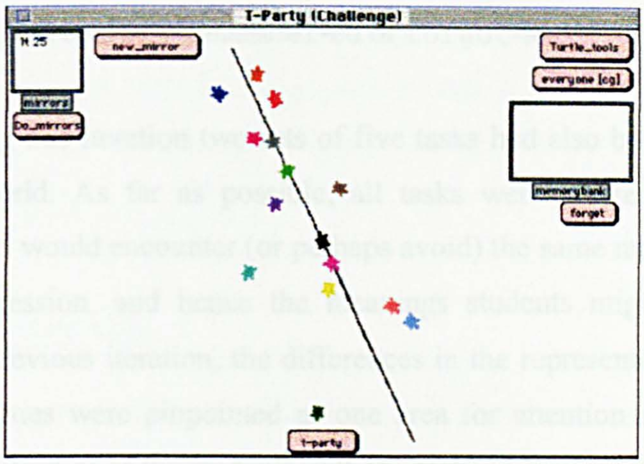
Epistemological emphasis: It was expected that student would use both intra and interfigural analyses as they applied and evaluated their own formalisations during this task. This task was also intended to emphasise the idea of the plane consisting of a set of points/turtles.

Reflect the set of points on your screen in the line shown.



Task 5 DEG: P-party task

Reflect the set of turtles on your screen in the mirror line.



Task 5 MTG: T-party task

5.2.2.3 The tools and tasks of the microworlds at the end of the second iteration

In the first iteration, the intra-inter distinction had been prominent in the design of tasks and tools. This concern also permeated this second iteration, but additionally issues relating to the formalisation of functional relationships came further to the forefront. To support students in this aspect, an important design decision was to make more visible, through task and tools, the symbolic texts conveying the theoretical descriptions of the corresponding visual productions in both microworlds.

The final microworld kernels: The DEG kernel itself was not changed during the second iteration. It included the original tools and interface features of the Cabri 1 system plus the two `compass` and three `angle carry` tools previously described. The MTG kernel, on the other hand, underwent a series of developments. The complete set of communication tools included the measuring tools `distance`, `towards` and `toheading`, tools for hatching new turtles, either on top of existing ones (`hatchhere`) or where the paths of two existing turtles intersect (`meet`), and tools to `flip` turtles or `swop` the orientation of turns in given commands. In terms of the interface features of the MTG kernel, textboxes were provided to display on screen symbolic records of the actions of particular turtles as well as to record and display variables (that could be `remember`-ed or `forgot`-ten)[11].

Tasks-sets: During this iteration two sets of five tasks had also been developed, one for each microworld. As far as possible, all tasks were created so that in their resolution students would encounter (or perhaps avoid) the same mathematical issues, though their expression, and hence the meanings students might construct were different. In the previous iteration, the differences in the representations of intra and interfigural properties were pinpointed as one area for attention in the comparison phase. From the activities of the students with the tasks as they were developed in this iteration, interacting with and expressing functions emerged as a second area in which to look for differences in the kinds of meanings students were constructing.

5.3 Iteration 3: Pedagogic Structuring

Of the various components of the learning systems as listed in *§4.2*, one aspect yet to be developed was the teaching interventions through which the two instructional approaches would be structured. Of course, as students had interacted with the microworld, the researcher had made numbers of teaching interventions, but it was during the third iteration that this aspect became the focus of more systematic attention. This section describes the development of teaching interventions of two forms: *global structuring*, the organising of what have been termed general teaching episodes and *local structuring*, which related to teaching interventions made while students worked on the computer activities.

In common with the other research activities conducted during the design phases, all aspects of the pedagogic organisation of learning systems were informed both by theoretical considerations and empirical observations of students' engagement with the systems in development. From the theoretical point of view, this iteration aimed to devise sets of interventions consistent with the two alternative instructional

[11] DEG and MTG tool definitions are available in Appendix 2.

approaches of filling-inwards (FI) and filling-outwards (FO) as described in *§2.4*. To test the global and local interventions, two student pairs worked through the entire learning sequence for the DEG learning systems, one pair following a FI approach and the other a FO one and another two pairs worked in the corresponding MTG systems.

5.3.1 Global structuring

An important component of all learning systems was the general teaching episode, which had the aim of emphasising knowledge of reflection in terms of institutional mathematical practices. FI approaches are characterised by the introduction of ideas in their general form from the outset of learning activity. In FO approaches, on the other hand, institutionalisation of knowledge follows from students' activity. So, the first difference between the FI and FO learning systems was the positioning of a general teaching episode (of 45 minutes duration). In the FI systems, this teaching episode would occur before students started on the five microworld tasks and in the FO systems after the tasks had been completed.

Remembering that one aim of learning systems was that they would encourage learners to develop views of reflection as a function, a question that emerges is what aspects of the function concept should be emphasised for such a view to be meaningful to students. Functions can be expressed in a dynamic form, as, for example, processes that map an element of the domain (independent variable or input) onto an element of the range (dependent variable or output). Functions can also be expressed in a more static form as relationships or correspondences between the two sets, domain and range (Sierpinska, 1992). In either form, in plane geometry, the group of isometry transformations can be thought of as functions in which the domain and the range are the plane itself. If the plane is conceived as an infinite set of points, then the independent and dependent variables of transformations are also points.

The challenge in designing the teaching episodes was to discuss the idea of mappings of the plane with students in ways that stressed its connection with what they already knew about reflection. To this end, three foci for discussion were isolated: co-ordination of interfigural and intrafigural properties; function as relationship (static view of function) and as transforming (dynamic view); and meanings for planes and their elements.

5.3.1.1 Filling-inwards teaching episode

The design of the general teaching episode for the FI systems involved the development of models to help students to connect their personal knowledge of reflection to the intended knowledge. Throughout the teaching episodes, four teaching strategies types were adopted, corresponding to those outlined in *§2.4.1*: *exposition* (presentation of models using mathematical voice), *leading* questions (encouraging students to use general and precise terms), *staging mistakes* (drawing attention to inconsistencies or errors) and *clashing* (provision of different valid representations of the same relations). In this section, the models presented in relation to each of the three teaching foci are described.

Focus on planes and their elements: To stress the two-dimensional notions of planes, they were described as spaces in which there was no up and no down. The metaphor of a flatland was used to introduce the idea of two-dimensional surfaces. In the general teaching episode, activity in flatlands was confined to a sheet of transparency and in the computer activities the computer screen represented the plane. The flatland (or plane) as well as any objects in it were described as being made up entirely of points.

Focus on functions: To model a transformation of the plane, students were shown an example of a plane first in its pre-image state and then its image after reflection in a given axis. Pre-image and image were initially presented separately to draw attention to the input-output relationship between the two versions of the plane, and to emphasise the idea of plane as both domain and range of the transformation, then the geometrical configuration made up of both pre-image and image was considered to focus on the relationships between the two.

Function as relationships: Different relationships between corresponding parts of pre-image and images were discussed with the students, starting with intrafigural relations and comparisons between the two flags and moving on to interfigural properties.

Function as transformation: A more dynamic image of function was also presented, as the researcher added various constructions to a second pre-image in order to produce its reflected image. The strategies included were three common "erroneous" strategies as identified in previous research as well as in the interviews and tests (reflecting in a vertical axis instead of the slanted axis shown, translating or rotating the pre-image). Additionally, two strategies for constructing the correct image were demonstrated both of which applying the same constructions to all the points shown in the pre-image using the same construction procedure and illustrating different interfigural properties (construction of equal perpendicular distances or two pairs of two equal angles either side of the axis).

Focus on co-ordinating intrafigural and interfigural properties: Students were also introduced to the idea of checking that properties associated with reflection that had not been used in the construction process were present in the final construction. This included attention to intrafigural properties, i.e., checking that the pre-image and image flags are congruent, or checking other interfigural properties.

5.3.1.2 Filling-outwards teaching episodes

The structuring of the FO systems did not include the presentation of "ready-made" models for reasoning about the intended knowledge. Instead the aim was that students would construct their own models of the situations described in the computer-tasks and that these models could serve as the basis from which they could *reinvent* models for reasoning about objects of reflection. The teaching strategies adopted in FO systems involved using the students' *voices* to re-express (and perhaps rethink) the intended knowledge from the researcher's *perspective.* It also involved the initiation of the strategies of *matching* (identifying and evaluating identical or overlapping solution approaches) and *contrasting* (identifying and evaluating different approaches to task solution).

The FO teaching episode addressed the same three teaching foci as its FI counterpart, but the order in which these were addressed was slightly different.

Focus on functions: The dynamic and static representations of functions were emphasised in opposite order in the FO as compared to FI systems, with students first negotiating mapping a pre-image onto its image and then asked to focus on relationships between input and output.

Function as transformation: In the first activity the students were asked to transform a given pre-image by reflection. There were also shown three incorrect images and asked to explain what had gone wrong during their construction.

Function as relationship: Students were then shown a pre-image and image which were superimposed to produce a symmetrical configuration. Students were asked to add its properties.

Focus on co-ordinating intrafigural and intrafigural properties: In the FI systems the researcher had drawn students' attention to particular elements of the figure. In the FO system this was not done and instead students were only asked to identify as many as possible of the properties of the symmetrical figure.

Focus on planes and their elements: No student models that could be considered analogous to the notion of the plane as a mathematical object were identified in the interactions of students during the specific teaching episodes, making it difficult to know how to encourage them to address this issue in a way consistent with the FO approach of building from students' inventions. In the end, additional points were added to the geometrical objects of the pre-image to ascertain how students treated these points.

5.3.2 Local structuring

Local structuring refers to the interventions made by the researcher as the students worked on the computer-based tasks. Regardless of instructional approach, the intention was that students would be in control of their own solution processes, making decisions and following directions of exploration that they chose for themselves. In practice, there were a number of different occasions when it seemed appropriate for the researcher to intervene in all four systems.

One important researcher-initiated intervention was the *introduction of new tools*, planned *a priori* (see task descriptions in *§5.2.2.2*) and which occurred either at the beginning or during computer-based activity. In FI systems, the tools were introduced in ways that attempted to emphasis their connection to aspects of the intended knowledge and in particular stressed geometrical objects as sets of points or turtles. So, for example, the `compass` tools in the DEG kernel were explained as tools that construct the set of points with a given distance from a given point. In contrast, in the FO systems, emphasis was on encouraging students to connect the empirical effects of a tool with their own knowledge, so, when the `compass` tools were demonstrated in FO systems, students were asked to come up with the own descriptions of their output.

When students requested information about the task or when it was deemed necessary to intervene because students had not understood the aim of the task or their activities had led them to a dead-end, then, regardless of instructional approach, the researcher's interventions took the form of open questions intended to ensure the responsibility for task resolution remained with the students. Additionally, where students did not spontaneously engage in activities to justify the constructions they made or their reasons for believing constructions to be correct were not obvious from their interactions, requests for justifications were made by the researcher.

One other intervention issue related to tool execution. No difference between FI and FO system was adopted in relationship to what were judged as essentially technical problems (for example, order of selection of inputs to DEG construction tools and syntax issues in MTG)[12]. In a technical intervention, the researcher helped the students resolve the problem in question in a direct manner.

5.3.3 The learning systems at the end of the third iteration

With the definition of the content and form of the global and local structuring, the four learning systems were considered ready to be brought to action in the next phase.

5.4 Summary of design phase

The design phase can be summarised by considering the findings emerging from three different activities: the probing of students' knowledge of reflection as expressed in paper and pencil setting; differences in the representation of the objects of reflection in the two microworlds; and the development of different strategies of intervention.

In terms of students' knowledge of reflection, the tests and interview responses elicited patterns of responses that correspond to those reported in other studies involving similar items. To summarise, students engaged in little mathematical analysis of an interfigural nature and focussed their attention instead on the relationships between internal properties of the pre-image and image. This analysis led to the construction of image figures as congruent to the corresponding pre-image figures, and located at the same distance from the axis. Different ways of interpreting the conservation of distance were connected to the production of images that were sometimes suggestive of isometric transformations other than reflection in the given axis, although many of these could actually be revisioned as two-dimensional representations of three-dimensional cases of reflection. Since the action and tools through which students concretise reflection (and especially mirrors) also bring to mind three-dimensional situations, it seems reasonable to propose that if students'

[12] The classification of a syntax problem as technical is somewhat relative to the mathematical objectives of the activity. For example, the order of selecting points determining an angle was treated as a technical problem in this study, if the angle concept had been the focus of the study, a rather different position might have been adopted.

meanings are to evolve in mathematical directions connected to plane geometry, some way of confining their activities to the plane is desirable.

Furthermore, by emphasising situations in which students' constructions might be considered correct, rather than looking for misconceptions or errors, the interview analysis, in particular, suggested a different way of interpreting students' evolving knowledge than that adopted in previous studies. Rather than involving the gradual build up of mathematical properties of reflection, in which students learn to co-ordinate the relationships amongst one type of variable at a time whilst ignoring others (as suggested by Küchemann (1981), for example), perhaps even to the extent that one aspect creates obstacles to the apprehension of others (as Grenier (1988) argues), the analysis during the design phase supports an alternative model of learning. The students brought to reflection activities complex sets of knowledge drawn from experiences inside and outside of the mathematics classroom. For this knowledge to evolve into the mathematics of reflection, students needed to accept and describe a model analogous to particular aspects of the reflection process as they know it. As they do this, mathematics will not only be abstracted out of, or even within, experience. It will also be used as a way of concretising experience.

The microworld tools and tasks and the teaching interventions were developed with this view of learning in mind. Hence, tools were developed to help students express what they already knew about reflection, as well as to afford the representation of this knowledge in new ways, which in turn might come to be associated with extensions in meaning. That is to say, the external resources were developed with students' internal resources in mind. Every attempt was made to construct the tasks for use with the two microworlds so that the new conceptualisations of reflection were equally possible and equally necessary in both. Nonetheless, there were epistemological differences between the microworlds in terms of the representations of intra and inter relationships, formalisations of functions and conceptualisations of the plane they made possible.

For example, the DEG microworld incorporated a tool (`symmetrical point`) that suggested an interfigural relationship between a pre-image point and its image, and displayed this relationship so it could be explored visually. This made it easy for students to see how some screen objects could be defined to depend on each other. Ironically, however, intrafigural relationships were less accessible for construction, inverting the order of intra/interfigural relationship as suggested in the paper-and pencil-context. While students experienced a strong visual portrayal of functional dependency, any corresponding symbolic representations were not paramount in DEG interactions.

In the MTG microworlds, the relationships between intra and interfigural analysis also diverged from the paper-and-pencil context. With its geometry based on communicating turtles, there was a blurring of the distinction between the intra- and interfigural and students were able to express abstractions about the relationships between as well as within pre-image objects and their images. While the tools of the microworld as presented to the students did not convey such a strong sense of functional dependency as the DEG tools, there was some indication that they supported a synergy between the visual objects constructed on the screen and the symbolic expressions that described them. The impact of these epistemological differences has been touched upon in the description of students' reactions to the developing tools. It will form a more central focus of the next chapter.

In terms of design principles, the development not only of the microworld tools and tasks, but also of the different teaching interventions has pointed to the need to provide external resources which enable learners to juxtapose formal expression with experienced activity. More than this, to be meaningful to students, new ideas are best presented in ways that resonate with activities that constitute part of the learners' identities outside of mathematics classroom and with activities which matter to them. For example, the `meet` tool developed in the MTG system matched an internal resource students used to think about points of intersection, but it also connected to ideas about two beings running into the arms of each other, a very human image of an intersection point.

The devising of appropriate interventions also represented an important activity of the design phase. Where models of general mathematical ideas were to be introduced to students, attention was again given to how to express general objects in ways the students would be able to concretise. Fundamental in both approaches was that the control for constructing solutions remained with the students during the microworld activity, so that they were given the chance to express their own knowledge using the means of expressing available in the DEG and MTG systems. Students' reactions to the instructional approaches received relatively little attention in this chapter, as rather fewer students took part in the trialling of this aspect of the system. They will be addressed in the following chapter.

Chapter 6

The comparison phase:

Systems-in-action, thinking-in-change

> "Every turtle has its own reflection turtle with the same distance away from the mirror and the same angle, except for lefts and rights"
>
> Aimee (MTG-FI, session 5)

The last chapter presented the design phase and described the iterative process through which the learning systems were developed. This chapter concentrates on the four systems in action. It focuses on evolutions in the ways students expressed the mathematical relationships of reflection and how these expressions related to the evolutions that the tool-sets, the tasks and teaching interventions were themselves undergoing. The chapter is divided into three sections. The first introduces the students and how they were chosen on the basis of their responses in the first activity of the learning system, the paper-and-pencil test. The second looks at the computer constructions built by these students during microworld activities, analysing how these were differentially conceived and expressed according to the mediational means of the four learning systems (microworld tools and instructional approaches). The third section considers the learning systems activities beyond the microworld and what happened when students returned to paper-and-pencil work after their computer-based interactions.

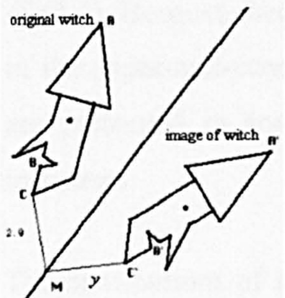
6.1 The students and their paper-and-pencil mediated views of reflection

The first activity was common to all four learning systems and involved the completion of the final version of the paper-and-pencil test. The complete test is presented in Appendix 3. In its final form, it was divided into five parts. The first part involved students in producing a written description of everything that they knew about reflection and reflective symmetry. The second part included six construction items, asking students to sketch images of various figures under reflection presented on squared paper. The third part presented two multiple-choice questions in which students were to select, from a variety of possible representations of a pre-image and its image by reflection in a given axis, those they believed to be correct. In each item, dotted segments, angles and their measures illustrated the process used to construct the image.

The fourth part of the test also involved multiple-choice selections. Students had to select the correct responses from sets of three statements about the generality of properties identified in a specific reflection representation. Finally, presented with two points, students were asked to construct the missing axis of reflection by which one point could be mapped onto the other and to describe how they had completed this construction.

Apart from the fourth part of the test, which had been inspired by activities trialled during the development of the microworld tasks, the test items were similar to those used in previous research studies (*§3.3.3.1*) and the design phase (*§5.1.1* and *§5.2.1*). The work around the microworld activities had suggested that one aspect of students' thinking that had not been addressed through the existing paper-and-pencil items was whether students could use information about the properties of the pre-image to deduce properties of the image, when this involved looking beyond the particular measures of the pre-image presented. If they were shown an intrafigural property of the pre-image and asked to consider it as the pre-image varied, could they deduce the corresponding relationship of the image? And what if the properties given highlighted an interfigural property?

It was in order to obtain data relevant to these two questions that the fourth part of the test was devised. It comprised four items. An example is presented in Figure 6.1. In each item, students were presented with the same visual representation of a pre-image figure, axis of reflection and image figure, but a different property highlighted. Along with each item, three conjectures related to the corresponding property of the reflective image were also presented and students had to choose the conjecture they believed to be correct.



Pauline, Queenie and Rob measured the distance CM (from the witch's chin to a point on the line of symmetry).

They found that the distance is 2.9 cm.

Pauline says that the length y is always 2.9cm, and that even if the original head or the line of symmetry is moved it will still be 2.9.

Queenie says that you can't say what length y is without measuring it.

Rob says that length y is 2.9 in this picture and that if you move the original head or the line of symmetry, the measure will change but the two lengths (CM and C'M) will still be the same.

Which of them do you think is right? (tick below)

Pauline | Queenie | Rob | Don't know

Figure 6.1: An item from the fourth part of the paper-and-pencil test

One of the conjectures suggested that the corresponding property would be equal to the marked property, and that the measure would remain invariant regardless of the position of the original figure or the axis (*equal and invariant*). Pauline's conjecture in Figure 6.1 gives an example of this form. In the second conjecture, it was proposed that the property of the image was *unknown*, and that it could be obtained by measuring (see, Queenie's statement in Figure 6.1). Rob's conjecture presented in Figure 6.1 is of a third form, in which the property is described as *equal but variable* – the property of the image will have the same specific measure as the original and will vary as the original is varied so that both properties always have the same measure.

Eighty-eight students from four Year 8 classes completed the paper-and-pencil test. The tests were analysed with two objectives in mind: to construct a picture of the complete sample, so that the six students participating in the remaining activities of the learning systems could be chosen as representative of the entire group; and to identify patterns of responses indicative of particular student types, so that each group would include students with equivalent *initial profiles*.

6.1.1 Overall picture of student responses

In general, students' responses to the paper-and-pencil items of the test followed the patterns identified during the design phase. An analysis can be found in Appendix 1 *(§A1.2)*. Because there had been no items equivalent to part four of the final version of the paper-and-pencil test in previous test versions, students' responses to this part are presented in some detail below and followed by a summary of the overall responses.

The distributions of students' choices according to conjecture forms are displayed in Table 6.1. The number of students who missed out these questions, or indicated that they didn't know how to respond (between 22% and 26% of the students) indicates that, despite writing descriptions and producing drawings in which equality of measures seemed a salient property of reflection, not all students applied this idea to make sense of the statements, or even to deduce that the required measure would be equal to the one that was given.

Students had more difficulty in making this deduction when the statement related to angle measures than lengths of segments, with only 38% of students in total responding that the angle in the first question would measure 47° and 17% recognising the perpendicular property as invariant when presented with item 4. The third item, which concerned the distance between corresponding points of the pre-image and image figure and a point of the axis, received the most correct responses (44%). It was somewhat surprising that fewer students felt that the distance between the mouth and chin on both figures would be equal and invariant (32%). This may indicate that some students did not, even before any microworld interaction, see figures as fixed shapes, for which movement would imply picking up the whole figure in its entirety and placing elsewhere, but as variable objects with changeable dimensions.

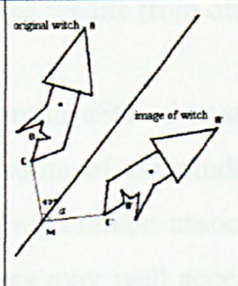
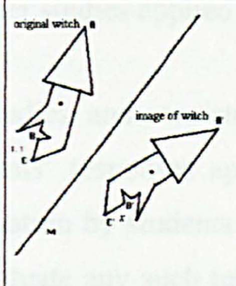
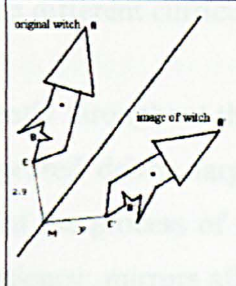
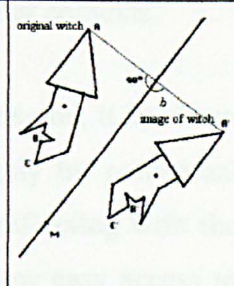
	Item 1	Item 2	Item 3	Item 4
	 *Property:* Angle to axis	 *Property:* Distance between points of figure	 *Property:* Distance to axis	 *Property:* Perpendicular distance to axis
	No. of students n = 88 (% in brackets)	**No. of students** n = 88 (% in brackets)	**No. of students** n = 88 (% in brackets)	**No. of students** n = 88 (% in brackets)
Unknown	32 (36%)	19 (22%)	19 (22%)	24 (27%)
Equal but variable	27 (31%)	22 (25%)	39 (44%)	28 (32%)
Equal and invariant	6 (7%)	28 (32%)	9 (10%)	15 (17%)
Don't know/ no response	23 (26%)	19 (22%)	21 (24%)	21 (24%)

Table 6.1: Distribution of students' choices according to conjecture form for the four items involving the identification of properties.

Summary

Figure 6.2 presents a description of reflection and reflective symmetry intended to characterise the kind of response that might be expected from one of the Year 8 girls who took part in this study.

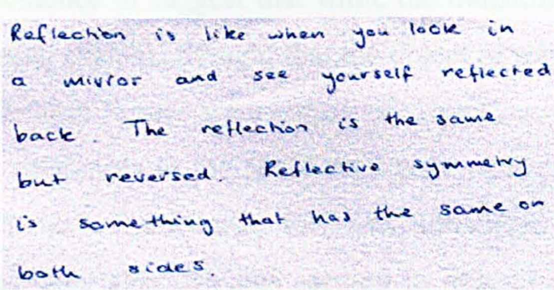

Reflection is like when you look in a mirror and see yourself reflected back. The reflection is the same but reversed. Reflective symmetry is something that has the same on both sides.

Figure 6.2: A description designed to show the meanings commonly expressed by students

In actions within paper-and-pencil contexts, the knowledge expressed within such descriptions was operationalised in ways that generally resulted in the construction of images in which the form and size of pre-images are preserved along with some distance from the axis. These results were obtained in every iteration of this study and also match the results from other studies applied in different curricular contexts.

Also in common with other studies, and consistently throughout this one, it has been found that some of the students' responses appeared driven largely by perceptual concerns. The common association by students of the process of reflecting with the use of mirrors may well accentuate any such tendency: mirrors allow easy access to images under reflection. But this is not the whole story. Test responses of other students seemed motivated by theoretical concerns rather than perceptual ones, with the conditions for accepting a reflective image being its congruency to the pre-image and the presence of (not always corresponding) equal distances to and angles at the axis. Results hence suggest a fragmentation of knowledge, with perceptual concerns not necessarily synthesised with theoretical counterparts and differences between students in which aspect dominated.

When students attempted to construct reflections, they operated on figures, either in their entirety or as a collection of segments and defining points. There is no evidence to suggest that students ever treated figures as homogeneous sets of points and, indeed no reason why they should enact such a treatment. However, it does not seem accurate to describe their activities as limited to the treatment of flat-fixed shapes, that might be slid, rotated or flipped, as suggested in some of the research literature reviewed (*§3.3.3.2*), although this may have been behind the responses of some.

There is also some evidence to suggest that while the transformation was applied to figures or their elements, students thought that the object of transformation was some larger whole in which these figures were contained. However, the only way students had to articulate any references to the space they were acting upon was through its divisions into two "sides". Sometimes, this was associated with applications of reflection to only one side, although this seemed to depend on the particular pre-images involved.

Whether students were thinking in terms of one figure or a larger space, the results can be interpreted as suggesting that it is the production of a complete symmetrical configuration which characterised students' activities, rather than the transformation of pre-image to image. That is, the outcome consists of both pre-image (one side) and image (the other side). For this reason is not surprising that intrafigural analyses dominated.

The consistency of these results, throughout the study and in relation to other studies, confirms the relevance of learning systems aiming to encourage students to extend their mathematical analysis from intrafigural to interfigural perspectives and to add to the idea of symmetry as a property a view of transformation as a mapping. The responses to the fourth part of the test suggest that students might also benefit from activities involving reasoning with the general properties as well as specific representations of them.

It is important to reiterate that in this formulation the evolution of knowledge is not conceived as a transition from one way of thinking to another more sophisticated way, neither is it seen as a quantitative accumulation. Rather evolution is seen as the connection of qualitatively different ways of thinking about the same concept, which extend the collection of relationships that might be made concrete in a particular situation as well as the extending range of situations in which these relationships might also be concretisable.

6.1.2 Selecting the four groups of six students

Having analysed patterns in the overall sample the next step was to decide on how to compose the four groups of students. Following from the discussion above, it was decided to try to make use of the analysis of overall responses to classify students into different types.

First, it was decided to identify students whose response profiles could be considered as typical of the sample as a whole. In addition to typical students (type I), it was also possible to isolate three other student types of research interest: type II were *perceptual* students, whose responses indicated a consistent tendency to construct and recognise images using perceptual rather than theoretical judgement – these students did not make any of the relationships between pre-image, axis and image explicit, but seemed to know what they should look like; type III were *theoretical* students who tended to emphasis theoretical properties over perceptual concerns – they already made use of some properties between pre-image, axis and image but were less aware of how the image should look; and type IV, students who could be placed at an *intrafigural extreme*, as they appeared have access to neither perceptual or theoretical resources that enabled them to distinguish a reflective image from one produced by any other isometric transformation, accepting and constructing congruent images of various locations.

The aim was that tracing the trajectories of the three more extreme types alongside the typical students would open multiple windows onto how students' internal resources afforded or constrained their interactions with the tools, tasks and teaching interventions in different ways.

Table 6.2 introduces the six participant of each learning system. The initial profiles of the twenty-four selected students and a description of how the four different types were distinguished in terms of these profiles are available in Appendix 4.

Type	Profile	DEG-FI	DEG-FO	MTG-FI	MTG-FO
Type I	Typical responses on all questions	Rhea	Sita	Hadley	Laurel
Type I	Typical responses on all questions	Anita	Rebekka	Lizzie	Candy
Type II	Visual images approximately correct	Suzie	Anju	Alissa	Prija
Type III	Theorectically motivated responses	Sharmila	Seema	Lorna	Jodie
Type IV	Images under various different isometries constructed or accepted	Christie	Maia	Helen	Sophy
Type IV	Images under various different isometries constructed or accepted	Elaine	Kylie	Aimee	Kerry

Table 6.2: The participants in the learning systems according to profile types

6.2: Microworld tools, tasks and teaching in action

This section focuses on students' interactions in the microworld as the student-pairs within each of the four systems attempted to construct and validate solutions to the five computer-based tasks.

Chapter 4 describes the methods used to organise the data collected during the five computer-based sessions into process profiles. In this section, these data are further analysed in terms the evolutions of the system during microworld interaction. Because of the similarity in the strategies that emerged in systems incorporating the same microworlds, trajectories associated with the two DEG learning systems will be presented jointly (*§6.2.1*), followed by the trajectories of the two MTG learning systems (*§6.2.2*).

For each task, the analysis of microworld interaction will be presented in the following manner. First, the main strategies developed by students will be outlined; the second section focuses on between-pair variations around the main strategies; third, variation according to FI and FO instructional approaches will be examined; fourth, analysis of how the tools and tasks of the microworlds appeared to *constrain and afford* the abstraction and concretion of knowledge of reflection will be presented in terms of its properties and its functional aspects, movements between intrafigural and interfigural analyses and students' treatment of figures and planes. Finally, evolutions to the microworlds will be presented. A complete set of students' computer constructions (in visual and symbolic forms) is available in Appendix 5.

In addition to the microworld activities, all the students participated in a general teaching episode mediated by the researcher (described in *§5.3.1*). In the two FI learning systems, this occurred prior to the microworld activities. This meant that students within these systems were introduced to aspects of the intended knowledge of reflection before embarking on the computer-based tasks. The particular knowledge stressed by the researcher included the co-ordination of intra and interfigural properties, functional aspects as a reflection (from static and dynamic perspectives) and the idea that geometrical objects consist of infinite point-sets.

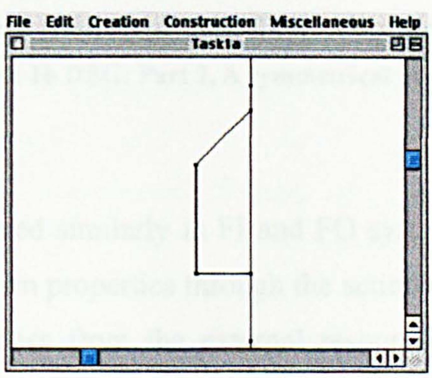
6.2.1 *Microworld interactions within the DEG learning systems*

Interactions around the each of the five DEG microworld tasks are discussed in this section.

6.2.1.1 Task 1: Completing figures with reflective symmetry in the DEG systems

This task was divided into two parts. Part 1 is shown in Task 1a DEG. It represents an item that the majority of students would be expected to solve successfully in a paper-and-pencil context and was considered a suitable vehicle through which students could further concretise geometrical properties with which they were familiar.

Complete the design on your computer screen by constructing the reflected shape.

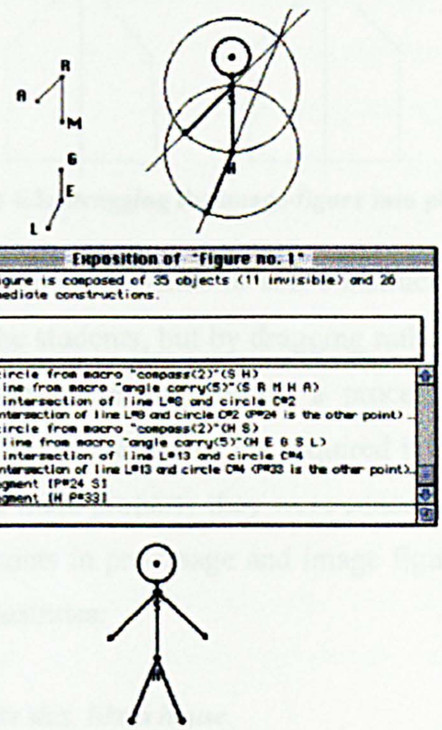


Check that your image is always correct by dragging the line.

Task 1a DEG: Part 1, Introductory figure

Part 2 presented students with a 'text' that defined the constructions used to build half a figure that could be dragged around the screen without messing up (Task 1b DEG). The aim was that students complete the figure by reflection.

Use the Cabri description of half a person to help in the construction of a complete symmetrical stick-person.



Check the figure remains symmetrical even when it is dragged.

Task 1b DEG: Part 2, A symmetrical person

Main strategies

Two main strategies evolved similarly in FI and FO systems. The first involved the expression of already known properties through the action of dragging. In the second, students identified properties from the external resources of the system and then formalised these into robust constructions. These strategies are examined below.

Expressing properties in soft constructions: making symmetrical drawings

All the pairs in both DEG systems tackled the first part of the task in the same way: adding three new line segments to the given figure and dragging them to produce a symmetrical figure (Figure 6.3 presents two moments in the production of the final drawing).

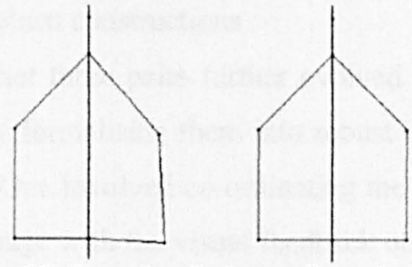


Figure 6.3: Dragging the image-figure into place

In DEG terms, the drawings had the status of soft constructions; the properties were purposely constructed by the students, but by dragging rather than through the use of the construction tools. They were engaged in a process of concretion; that is, theoretical concerns about the properties of the required image guided the empirical activity of the students. The main property they were concretising was that of equality in corresponding line segments in pre-image and image figures, as the extract below from one of the six pairs illustrates:

Anj: *It's easy, goes like this, like a house.*

Sit: *A point?*

Anj: *We could do a...line segment, use that point and then...about here. Yes. Are they the same length?*

Sit: *Mmm, and OK, now another one, to make the same as this one.*

(Sita and Anju, Type I-II pairing, DEG-FO)

None of the students was motivated to use the construction tools when their drawings became messed up as the axis was dragged. They could see that the figure no longer remained symmetrical, but all six pairs reacted to this feedback by simply adjusting their image segments until the whole shape looked symmetrical once again. In their activities at this stage they were expressing generality in action (see Mason, 1996; p.81) rather than building it into their constructions.

In Part 2, three pairs of students continued to use the strategy of constructing by eye. This involved adding two new segments, for the second arm and leg, then dragging these into place every time that any element of the whole construction was moved.

Expressing properties in robust constructions

The approaches of the other three pairs further evolved as they made properties of reflection more explicit by formalising them into robust constructions. Two different strategies were observed. One involved co-ordinating the symbolic description of the construction on the pre-image with the visual feedback obtained as it was dragged on screen; the other involved abstracting a geometrical property from visual feedback.

Coordinating visual and symbolic resources: This strategy made use of the `exposition` text (see Task 2 DEG) to support the formalisation of an equal angle relationship. The discussion of a pair in which one student dragged the figure on screen as the other attempted to interpret the text in the exposition is shown below (and their construction is presented in Figure 6.4). It illustrates how the strategy brought into play thinking about two different aspects of knowledge: dragging both emphasised the generality of particular properties and provided a visual representation of functional dependency:

Ani: *His arm moves when you move one of the points here...If I move this point up it moves up, the arm goes up and, if it goes down, the arm goes down too.*

Sha: *I think is got to be here, angle carry, that copies angles doesn't it....and there's another. What happens to the leg?*

Ani: *The leg is the same, I move the angle here and it moves there, see?*

Sha: *We need, if we could copy it here, for the other arm.*

(Anita and Sharmila, Type I-III pairing, DEG-FI)

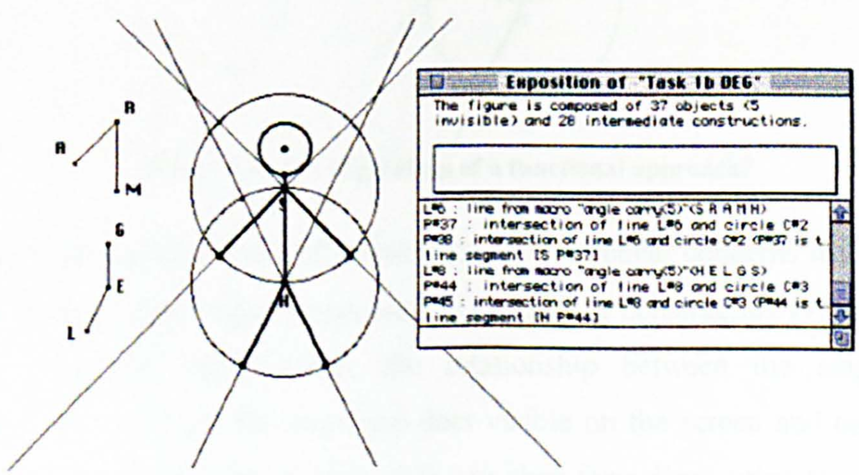


Figure 6.4: Locating properties in a symbolic description

Formalising an interfigural relationship: Interactions of one pair suggested their attention had shifted from intrafigural to interfigural analyses: they focussed on the position of an image point relative to its corresponding pre-image point. This pair did not use the `exposition` text, but their visual explorations led to the identification of a general property, which they represented using the `perpendicular line` tool as shown in Figure 6.5.

> After a short period of undirected dragging, Maia suggested to Rebekka that she slowly drag point *A*. They observed what happened, then discussed the required behaviour of the second arm.
>
> Reb: *If it's a reflection, then both arms should move.*
>
> Mai: *It will be ... move it again a minute.*
>
> As Rebekka dragged point *A* once more, Maia mirrored the movement of the dependent vertex (hand) of the existing arm, with a finger showing the location of an imaginary symmetrical point. She then placed a finger perpendicular to the body and continued to track the 'hand' vertex.
>
> Mai: *It's on a line going across here, where it crosses here. What's that called...90°?*
>
> (Rebekka and Maia, Type I-IV pairing, DEG-FO)

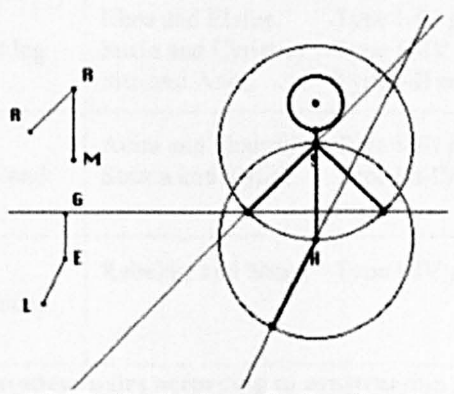


Figure 6.5: The beginnings of a functional approach?

Whereas in the definition of soft constructions, theoretical concerns had guided the empirical activity and dragging was used as a means of constructing properties, in the building of robust constructions, the relationship between the empirical and theoretical was reversed: the empirical data visible on the screen and on paper had helped students identify the property that was then formalised using the tools of the system.

Between-pair variation

Table 6.3 shows how the types of students were distributed among the three strategies used during the second part of the task. Both the perceptual (type II) students were among those who built soft-constructions only, a strategy that also emphasised perceptual concerns and treated a DEG diagram as a drawing that can be altered at will. This can be compared with a tendency in pairs containing the more theoretical (type III) students to go on to build robust constructions by using an `angle-carry` tool as a way of formalising the equal angle relationship.

Both theoretical and empirical considerations were involved in construction of both soft and robust constructions, but these computational objects differ in the level to which properties are made explicit and the emphasis on functional aspects – the tendency of the type III students to want to make properties explicit meant that they (and their partners) also engaged, more than those who built only soft constructions, with exploring how formalising DEG properties set up functional dependence relationships.

Strategy	**Student pairs**		
Soft-construction (adding segments for arm and leg and dragging into position)	Rhea and Elaine, Suzie and Christie, Sita and Anju,	Type I-IV pairing, Type II-IV paring, Type I-II paring,	DEG-FI DEG-FI DEG-FO
Robust construction (synthesising visual feedback and symbolic text)	Anita and Sharmila, Seema and Kylie,	Type I-III pairing, Type III-IV pairing,	DEG-FI DEG-FO
Robust construction (identifying a general relationship from visual feedback)	Rebekka and Maia,	Type I-IV pairing,	DEG-FO

Table 6.3: Distribution of student pairs according to construction strategy used for second task

However, it was not only in pairs containing a type III student that robust constructions were defined: the third pair who came up with a robust construction was a type I-IV pairing – a typical student paired with a student who had appeared to associate reflection with the production of any congruent image. There was nothing in their initial profiles that could be used to explain why this student pair should have adopted an interfigural perspective to this task. Instead it seemed to have been motivated by their interactions during the task; in particular, the dynamic behaviour of the half-stick person, along with the presence of construction lines on screen.

Between-system variation

There was no evidence to suggest relationships between the strategies used and instructional approaches. It might have been expected that students from the FI systems would have shown some awareness of interfigural as well as intrafigural properties during their interactions, since these had been explicitly signalled during the teaching episode, but this did not happen: the only pair who formalised an interfigural property had identified this during microworld interactions within the FO learning system.

The local structuring in FI and FO systems did not seem to be differentially associated with the ways students interacted with the task and interventions were required at the same point in both systems, with all students needing technical help the first time that they made use of the construction tools.

Constraints and affordances

During the first part of the task, students used their knowledge about intrafigural relationships to solve the task perceptually, focusing on the congruency between pre-image and image segments. Although movable segments provided students with a new way of constructing congruency, the knowledge expressed was similar to that which emerges when reflective images are drawn on paper.

The first images students produced were not the result of robust constructions, but this was not a cause for any conflict at this stage, as students had not yet connected to the idea of defining objects to depend on others. Even when the axis was dragged, it was not very difficult for students to adjust the second half of the figure into a form representing a symmetrical image. The drag facility therefore provided a way of expressing equal lengths without denoting the relationship between two line segments with reference to the particular formal system of DEG; that is, without formalising it. As in the paper-and-pencil activities, the students' goal was directed towards the visual products – each time objects were moved on screen, students responded as if faced with a new example to resolve, treating pre-images as multiple static instances rather than dynamic variable cases.

When it came to the second part of the task, the additional external resources, provided in the forms of symbolic text and construction lines on screen, enabled half of the students to engage in formalising activity. Two student pairs synthesised the visual with the symbolic forms, and this enabled them to identify from the visual feedback a property that they could connect with a construction tool listed in the symbolic text – intrafigural analyses of the internal relations of the desired symmetrical figure were associated with DEG tools in the given description. This helped students connect properties with visual images of geometrical dependency, whilst still working with a view of reflection as a property not a function.

One pair managed to identify a relationship on the basis of the visual feedback obtained as the given half-figure construction was dragged. In this case, the dependencies set up in the given construction seemed to have motivated the pair to focus on an interfigural relationship between input and output variables and only afterwards on the final symmetrical person that this relationship defined. In action, reflection was being treated as a function.

The provision of the half-figure construction hence not only afforded to some students a more explicit focus on properties, it also enabled a situated abstraction related to geometrical dependency: when construction tools are used, objects can be built to move together. With this in mind, the dragging facility could assume a new function: that of validating the robustness of constructions – a way to concretise geometrical dependency. Engaging with robust constructions also afforded evolutions in the ways students thought about figures: adding the view of figure as dynamic example to existing views.

In summary, moving from soft to robust construction was accompanied by an increasing explicitness about the properties used to construct reflective images, provided a means of seeing, discussing and formalising functional dependence and involved extending meanings for figures so that they could be seen as generic cases as well as particular examples. Not all students, however, made this move.

Figure 6.6 presents a diagrammatic representation of the passage of students through the learning system thus far. It summarises the knowledge presented by the researcher in the general teaching episode of the FI learning system as well as the knowledge expressed in the computer constructions built by the student pairs during the task.

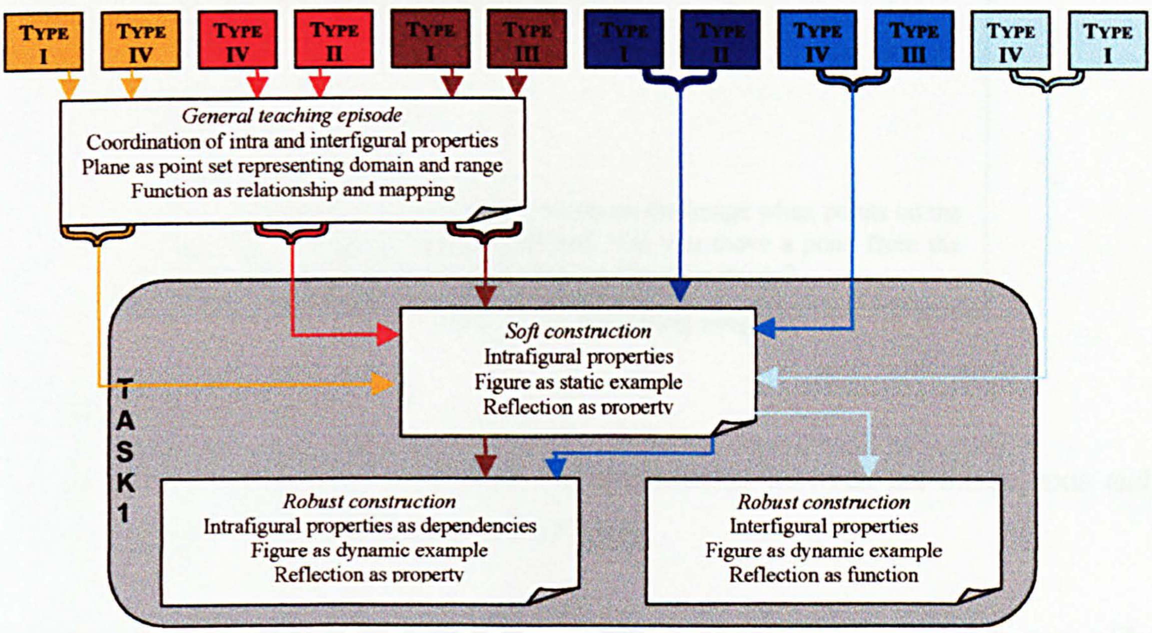


Figure 6.6: Student trajectories through the DEG learning systems to the end of first microworld task

Microworld evolutions

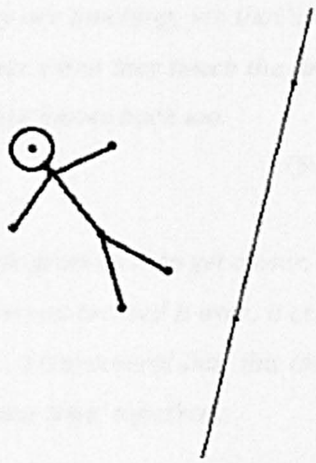
The DEG microworlds had been extended to include a stick-person figure, whose behaviour under dragging depended on the construction activities of the students. These constructions were used during the next task.

6.2.1.2 Task 2: Reflection as a relationship between image and pre-image

The aim of this task (Task 2 DEG) was that it would focus attention on relationships between rather than within figures. The first part of the activity involved the reflection of a figure located initially on one side of a line that divided the screen. The second involved attempting to make a pre-image point and its image coincide.

Construct the image of the stick-person by reflection in the line on your screen.

Check your method works for different positions of the person and the line.



Investigate what happens to points on the image when points on the line or original figure are moved. Can you move a point from the original stick-person so that it is on top of its image?

Task 2 DEG: Reflecting people

Main Strategies

The main strategies used to explore the relationships between pre-image, axis and image in the DEG systems are described below.

In contrast to the other construction tools, `symmetrical point` was readily appropriated by all student pairs in both DEG learning systems as they constructed the image-figure of the stick-person in the same way – applying the tool to each vertex of the figure in turn, then joining these points with line segments.

When points on the pre-image stick-person were manipulated, intra and interfigural aspects related to the movements were discussed and all student pairs identified that image points coincided with pre-image points when they met on the axis of reflection. The following two extracts illustrate how students described the relationship between elements of the pre-image, their images and the axis of reflection.

Suz:	*The arms are growing together*
Chr:	*This gets longer, making the other get longer too and...*
Suz	*...both points are moving towards the line.*
Chr:	*Now they are touching, yes that's what we were supposed to do.* ***They touch each other when they touch the line.*** *And moving this one back away...*
Suz:	*...the other moves back too.*

(Suzie and Christie, Type II-IV pairing, DEG-FI)

Sha:	*The heads grow as they get closer, now it's big, now its small.*
Ani:	*And when you crossed it over, it crosses over too.*
Sha:	*It's...this, I can control this, this one, moving this.*
Ani:	*At the same time, together...*
Sha:	*...gether.*
Ani:	*So did we make it go on top? It did, didn't it?*
Sha:	***It goes on top when it's on the symmetry line.***

(Anita and Sharmila, Type I-III pairing, DEG-FI)

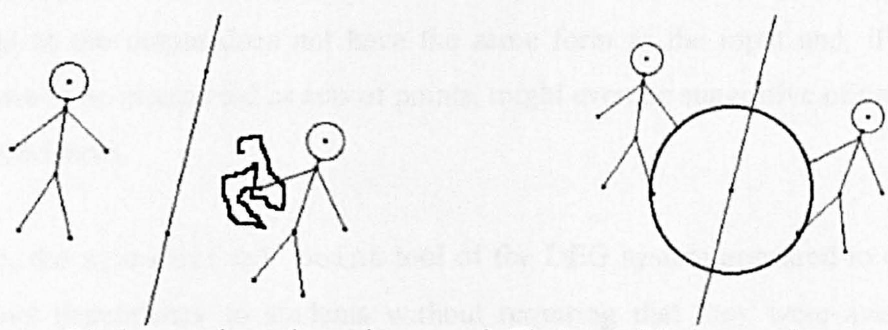
Evident in both these exchanges is the idea that the movement of the image point is controlled by the movement of the pre-image point, although the descriptions of this movement are not expressed using the formal language of the system.

The `locus` tool was introduced at the beginning of this task in an attempt to encourage students to describe behaviour of the screen elements in terms of the geometrical objects of the DEG system. Observing the locus of an image point as its pre-image partner was dragged (Figure 6.7a) did not help students express geometrical properties in a more explicit way, as the following example illustrates:

See:	*It goes ... all over the same.*
Kyl:	*Huh?*
See:	*Moves the same way as the other.*
Kyl:	*Higgledy-piggledy.*

(Seema and Kylie, Type III-IV pairing, DEG-FO)

And when the locus of the image point in relation to one of the points defining the axis was obtained, although students could see how the image point described a circumference (Figure 6.7b), this also did not lead to any explicit articulations of the equal distance property. Nor did students appear to engage with this feedback as had the pairs exploring the "*dancing*" dynamic figure of the design phase (described in *§5.1.2.1*). There were two differences that explain the relative lack of engagement. First, in their exploration of the dynamic figure they were searching for properties to construct. Here, students were exploring only. Second, in the activity of the design phase, students had found a way to connect the geometrical movement with movements in a human context. Giving another sense to the movement seemed to signal a greater investment in the task on the part of the students. The objects and relationship came to matter more to them than those observed and identified with the help of the `locus` tool during this task.



a: The locus of the image point as the pre-image point is dragged

b: The locus of the image point as a point on the axis is dragged

Figure 6.7: Recording the path of an image point

Between-pair variation and variation associated with instructional approaches

All twelve students, regardless of their types, had concentrated on the `symmetrical point` tool throughout this task and there was little to distinguish between the knowledge they used and expressed. Similarly, there were no obvious differences attributable to instructional approach.

Constraints and affordances

Taking the DEG students' constructions of the image of the stick-person as a whole, it is interesting to note the uniformity in construction. Each stick-person was reflected by applying the same operation to each of its vertices in turn. From the paper-and-pencil tasks of this study, as well as in previous research, this is a strategy that is relatively infrequently adopted when students produce reflective images on paper-and-pencil. The DEG `symmetrical point` tool constrained students to think about figures as defined by their (dynamic) vertices and to employ a strategy that can be aligned with a view of reflection as a function – based on relationships between the inputs (pre-image point and axis) and output (image point).

The application of the same method to construct the image point of each vertex of the pre-image was more or less automatic once students had seen the `symmetrical point` tool. Its ready inclusion by students into their constructions presents a stark contrast to the use of the construction tools in the first task. Why might this be the case? Might it be that, because of the one-to-one correspondence between independent and dependent variables, each point has one and only one image, which is another point? In tools like `angle carry` and `compasses`, the situation is different as the output does not have the same form as the input and, if lines and circles are to be interpreted as sets of points, might even be suggestive of one to many correspondences.

In short, the `symmetrical point` tool of the DEG system appeared to emphasise functional dependency to students without requiring that they were aware of the particular relationships behind it. It had been intended that the relationships would emerge through empirical activity – the on-screen data produced by dragging and by the `locus` tool – and its connection to what students already knew about reflection. Dragging did help students to communicate about the ways in which figures were related. Specifically, students were enabled to describe the movement of the image in relation to the pre-image, and all of them observed that a point and its image coincide on the axis. However, at this stage, any abstractions were situated in a world of moving bodies and not yet articulated using a geometrical symbol system.

Figure 6.8 below presents a schematic summary of the trajectories of the student pairs through the learning system thus far. It shows how the introduction of the `symmetrical point` tool allowed all students, regardless of their previous experiences, to build robust constructions without making any specific properties of reflection explicit. It encouraged students to treat figures as objects defined by a set of dynamic vertices and to enact reflection as a particular dependence relationship of image points on pre-image points.

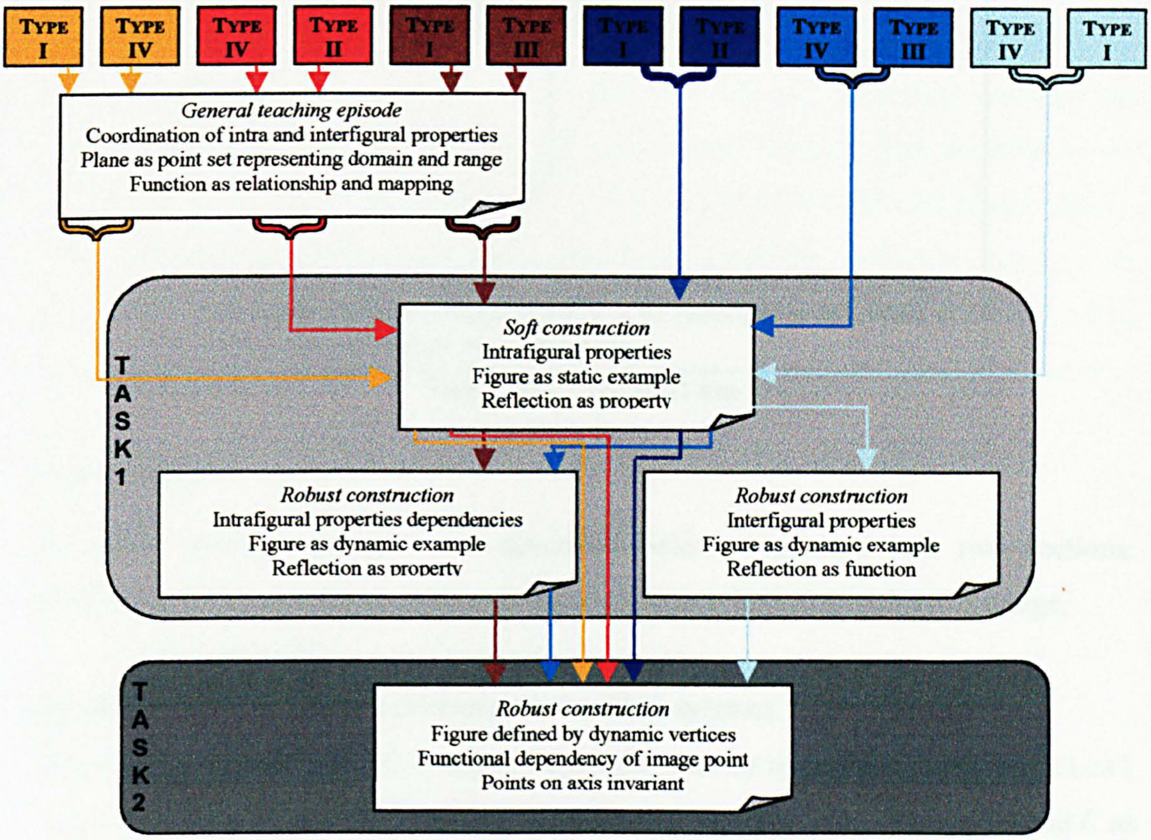


Figure 6.8: Student trajectories through the DEG learning systems to the end of second microworld task

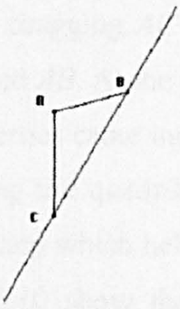
Microworld evolutions

The microworld tools did not evolve during students' interactions, but all students did construct robust computational objects from which they could see the results of reflection on points anywhere on the computer screen.

6.2.1.3 Task 3: From symmetrical figures to image points

The first task had been designed to emphasise relationships within figures and the second to highlight relationships between them. Generally speaking, this is what had occurred, although, as yet, not all of the students had formalised the intrafigural properties used in the first task and it was the dependency relationships rather than the between-figure properties that had been emphasised in the second. This task (Task 3 DEG) was intended to bring the focus back onto properties.

What kinds of quadrilaterals can be made by reflecting one point (A) of a triangle in a line passing through its other two points (B and C)?



Without using the `symmetrical point` tool, find different ways to construct another image point of A by reflection in BC. Write a Cabri macro based on one of these ways.

Task 3 DEG: The DEG kite

Main strategies

The main solution strategies are described below, organised into two sections: manipulating a dynamic quadrilateral; and from quadrilateral to point and image.

Manipulating a dynamic quadrilateral in the DEG systems

All six pairs in the two DEG systems began the task by applying the `symmetrical point` tool to point A of the triangle, using the line passing through points B and C as an axis of reflection (Figure 6.9).

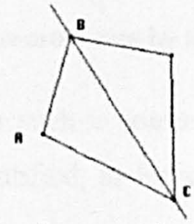


Figure 6.9: A quadrilateral symmetric about one of its diagonals

Particular types of quadrilaterals were attempted by dragging the vertices A, B and C as students engaged in the process of trying to soft-construct squares, rectangles, rhombi and parallelograms. Most pairs also checked whether their soft constructions had the necessary properties of the intended quadrilateral type, making them explicit either by adding measures or using the construction tools. In this way their interactions cycled between theoretical and empirical aspects: theoretical concerns guided the initial dragging activities; then other properties were identified from the examples generated.

As different quadrilateral types were attempted, students could see which properties held for all configurations of the symmetrical quadrilateral. For example, the square shown in Figure 6.10 was built by dragging *AC* so that it was vertical and had the same length as the horizontally placed *AB*. At the point at which the segments seemed to be equal, a number of other properties came into view and by adding constructions to illustrate them, and then dragging the quadrilateral again, it was possible to see which were particular to the square and which held for all symmetrical quadrilaterals. The construction lines in Figure 6.10 show the properties one pair identified as always satisfied: adjacent line segments and angles either side of the axis and a line perpendicular to axis and passing through point *A* and its image (originally constructed as the second axis of symmetry of a square).

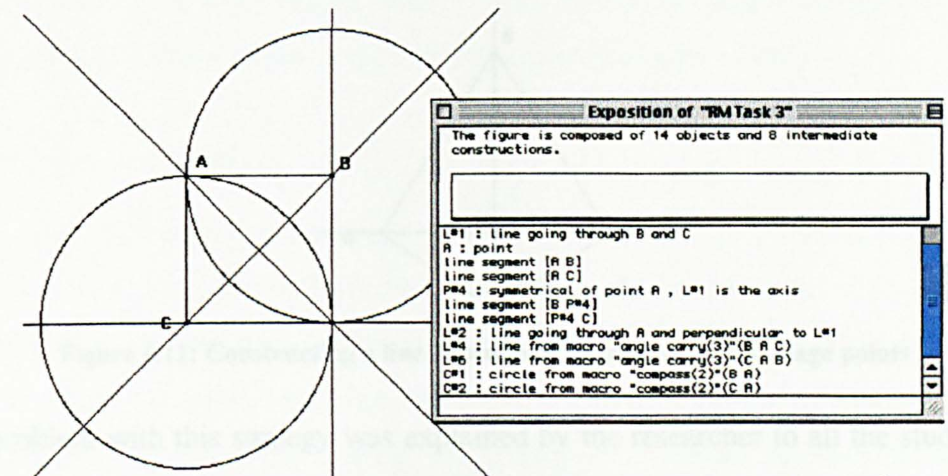


Figure 6.10: Verifying the construction by the use of construction tools[1].

Most student pairs did not isolate such a comprehensive set of properties as those shown in Figure 6.10, but all identified, at least, the two pairs of adjacent sides of equal length.

From quadrilateral to point/turtle and image

During the second part of the task, most pairs did not manage to formalise the properties identified in the quadrilateral in ways that would produce their own general

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[1] The points used within the `angle carry(3)` macro are not displayed in the `exposition` in the same order that they are selected. The students actually copied angles *ABC* (displayed `angle carry(3) B A C`) and *ACB* (displayed `angle carry(3) C A B`).

tool for reflection, although they all were able to enact properties sufficient to define a reflective image point.

An initial difficulty in the transition from focussing on the quadrilateral to constructing a relationship between two of its vertices was the use of a circular construction strategy, in which the output from the now "banned" `symmetrical point` was used to construct a line upon which students wanted to position their new image point (Figure 6.11). Although invalid as a construction approach, the line provided an external visual sign of the perpendicular relationship between the axis of reflection and the line joining pre-image to image point: it signalled one property of the correspondence between input and output, but without requiring its formalisation.

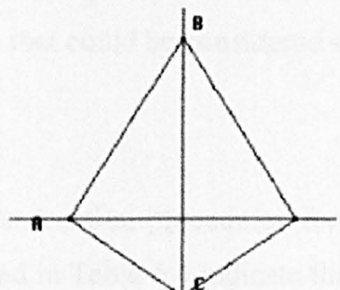


Figure 6.11: Constructing a line dependent on pre-image and image points

The problem with this strategy was explained by the researcher to all the students in both systems and an additional criterion for assessing constructions was imposed: that deletion of the original symmetrical point must not also result in the deletion of the pair's image point.

Following this intervention, the main strategy was to add a line that passed through the pre-image point *A* and was perpendicular to the axis of reflection. Some pairs used either basic lines or lines by two points, although most went on to replace lines positioned manually with lines constructed using the `perpendicular line` tool. Once they had the perpendicular line on screen, students knew that the image point should be located along this line at a distance to the axis equal to that of the pre-image point. Their focus was on describing the correspondence relationship between pre-image and image point:

They have to be on this line going straight across and have the same distance both sides.

(Suzie, type II student, DEG-FI)

I would say, symmetrical points can be joined with a perpendicular line and the measure along the line is equal both sides.

(Seema, type III student, DEG-FO)

In contrast to the first task, the majority of students no longer seemed entirely satisfied with an image point that required some manual adjustment. They wanted to build a robust construction and they knew which properties to include. What they did not seem to know was which DEG tool could be used to construct the equal distances they wanted – in a way that its equality was maintained under dragging – hence, ending up with a construction that could be considered semi-robust.

Between-pair variation

There were differences in construction procedures for image points that developed across the pairs. Data presented in Table 6.4 indicate that the two pairs containing the perceptually motivated type II students used no construction tools, whereas both the pairs in which the theoretically orientated type III students worked constructed the perpendicular property. Type I and IV students were spread out between the different strategies.

Image point constructions	Student Pairs		
Soft construction Equal distances from two points on axis	Sita and Anju,	type I-II paring,	DEG-FO
Soft construction Equal perpendicular distances	Suzie and Christie,	type II-IV paring,	DEG-FI
Semi-robust construction Equal perpendicular distances (distance by eye)	Anita and Sharmila, Seema and Kylie, Rhea and Elaine,	type I-III pairing, type III-IV pairing, type I-IV pairing,	DEG-FI DEG-FO DEG-FI
Robust constructions Two equal angles Two equal distances Equal angle and distance Equal distance and perpendicular line	Rebekka and Maia,	Type I-IV pairing,	DEG-FO

Table 6.4: Image point constructions of the student pairs

The continuing use of soft-construction by pairs containing type II students suggests that they were still treating their on-screen constructions as drawings rather than figures. This was especially true of the type I-II pairing from the FO system who were the only pair who did not seem concerned that their image point messed up under dragging. They also did not identify the perpendicular relationship and instead developed a soft construction based on the distance properties identified during the symmetrical quadrilateral activity. It involved the use of two line segments, now seen as distances from the axis rather than sides of a quadrilateral. A segment from B (labelled BA' in Figure 6.12, although the students did not actually name it) was dragged to have the same length as BA, then dragged again, carefully maintaining the same length, until CA' was the same length as CA. Finally the segments were hidden. Figure 6.12 shows three moments of the soft-construction process.

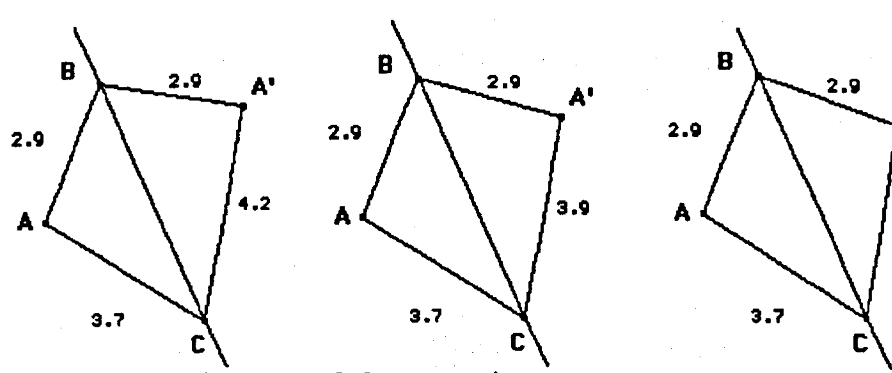
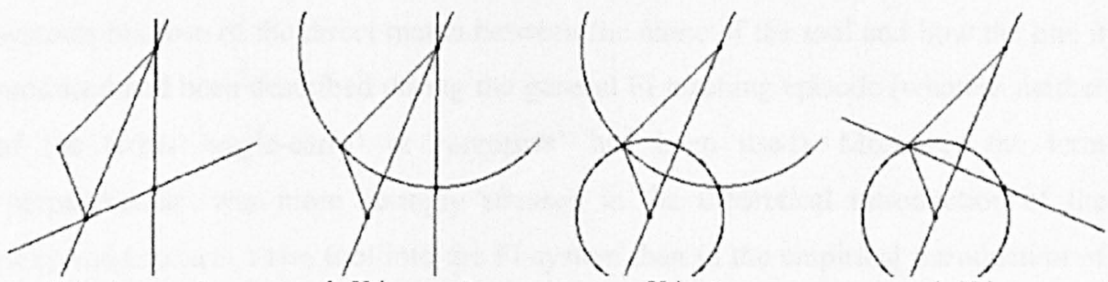


Figure 6.12: Soft constructing two equal distances

Finally, there was one pair (a type I-IV pairing) who adopted a strategy very different to the rest, constructing not one but four robust constructions of an image point. This pair not only had made use of the construction tools from their first encounter with the microworld, but had also made the most extensive use of the DEG tools during the first part of the task. Their image-point constructions evolved as a direct result of the way they had verified the properties of their symmetrical quadrilateral (shown in Figure 6.10 above). When the reflected half of this quadrilateral was deleted, all the construction lines used in their verification activities remained on screen, with five different objects intersecting at the location of the image point. The pair worked out that they could construct their own image point using intersection points based on any two of these lines. The four constructions they built are presented in Figure 6.13.



a: Using two `angle carry(3)` constructions

b: Using `angle carry(3)` and `compass(2)`

c: Using two `compass(2)` constructions

d:. Using `perpendicular line` and `compass(2)`

Figure 6.13: The four robust image point constructions defined by Rebekka and Maia

It seemed that, for these two students, the introduction to the DEG tools afforded an engagement in interfigural analyses from the outset of their microworld interactions and this changed the way they thought about reflection: their focus of the behaviour of the dynamic construction gave them a way of verifying when reflections were obtained. They were extremely motivated to come up with as many ways as possible of defining the same point – the only pair who became engaged in the construction of multiple methods – and what seemed to make this possible was their efforts to make sense of the construction tools in ways which went beyond getting the immediate task done.

Between-system variation

Looking back at Table 6.4 shows that all three students from the FI system used the equal perpendicular distance method in their image point constructions. Was there something about working in this particular learning system that had emphasised this method? There were no visits between computers during this activity; nor were properties discussed between the pairs in any overt manner. So were the students recalling and reproducing these properties from the general teaching episode? If this was the case, then it could be argued that the FI introduction prioritised a particular interfigural procedure, perhaps at the cost of suppressing other possible relationships. There was certainly more diversity in the properties enacted in the FO system. This does not explain, however, why the other construction model introduced in the FI teaching episode, the equal angles method, showed up only in the interactions of an FO pair.

It may have been that the perpendicular relationship was emphasised in the FI systems because of the direct match between the name of the tool and how the line it produced had been described during the general FI teaching episode (whereas neither of the terms 'angle-carry' or 'compass' had been used). Moreover the term 'perpendicular' was more strongly stressed in the theoretical introduction of the `perpendicular line` tool into the FI system than in the empirical introduction of the FO systems.

Turning to the researcher interventions during microworld interaction, apart from technical help, the researcher also intervened in both learning systems to encourage students to verify their constructions and justify that they represented valid solutions given the task demands. These interventions were made in the same way in both systems: the problem of circularity in construction was explained to FI and FO students alike and requests for justification posed as questions only in all systems.

Constraints and affordances

During this task, students' interactions suggested that they were evolving meanings in which the properties of reflection were becoming connected with its behaviour as a function. The `symmetrical point` tool, in combination with the dragging facilities, had encouraged students to distinguish between independent (pre-image) and dependent (image) points. The tool also had a data-generation role, providing students with access to visual evidence of reflective images, without calling on activity in three dimensions and without emphasising operations on only half the plane.

When the task changed from exploration to construction, students' focus also shifted from the quadrilateral to relationships between its vertices. The activity became that of modelling the behaviour of the `symmetrical point` tool. Instead of seeing points A, B and C as a triangle, point A became the pre-image point and the segments AB and AC its distance from points on the axis – intermediate objects rather than aspects of the required visual product. It may have been this that also prompted many pairs to add another intermediate object to the figure: the line that passed through A and crossed the axis at right angles.

For those who could describe this relationship as perpendicular, it was not hard to connect this with the DEG `perpendicular line` tool. Ironically, it was the second property necessary to complete the reflection function – equal distances – that the majority of students could not formalise in DEG terms. In their paper-and-pencil work, this property was made explicit far more frequently than the perpendicular one, but its familiarity is firmly connected with metric not geometric construction. Even though all the students had come across the `compass` tools in previous DEG activities or seen the circumference described by a symmetrical point as the axis is dragged, the role of circles in constructing equal lengths was not yet connected to the problem-in-hand. Perhaps the way that the DEG tools constrained students to think in terms of input and output hampered rather than helped: they wanted to output a point along the perpendicular line and, with their still very limited experience of geometrical constructions, most did not think of constructing another object that crossed the line at the required location.

One other general affordance of the DEG microworld relates to dragging. Dragging not only afforded the production of unlimited visions of reflective images, but had also encouraged students to see a point not as the highlighting of a static location, but as a dynamic entity used to represent any location on the screen – a generic point. With this meaning for points, it became more necessary to think about general methods rather than the production of specific visual configurations. The task of constructing a reflective image of a point was not just drawing its image point in the correct location; it was also connected with defining the relationship between a point and its image. This was evident in some of the verbal descriptions of the students, but also in their actions – because most did not manage to define robust construction, they had to physically reposition the reflective image if the pre-image, or axis, was moved and, as they did so, they enacted the same procedures each time. The view of function that appears to be emphasised was that of relationships between two sets (the static view as described in *§5.3.1*), rather than of the mapping of one set onto the other.

Figure 6.14 presents a schematic summary of the trajectories followed by each student pair to the end of this task. It shows that, by the end of this task, all students had given some attention to inter as well as intrafigural properties and, although not all had been able to make geometrical properties explicit, the majority had wanted to include some degree of functional dependency in their construction.

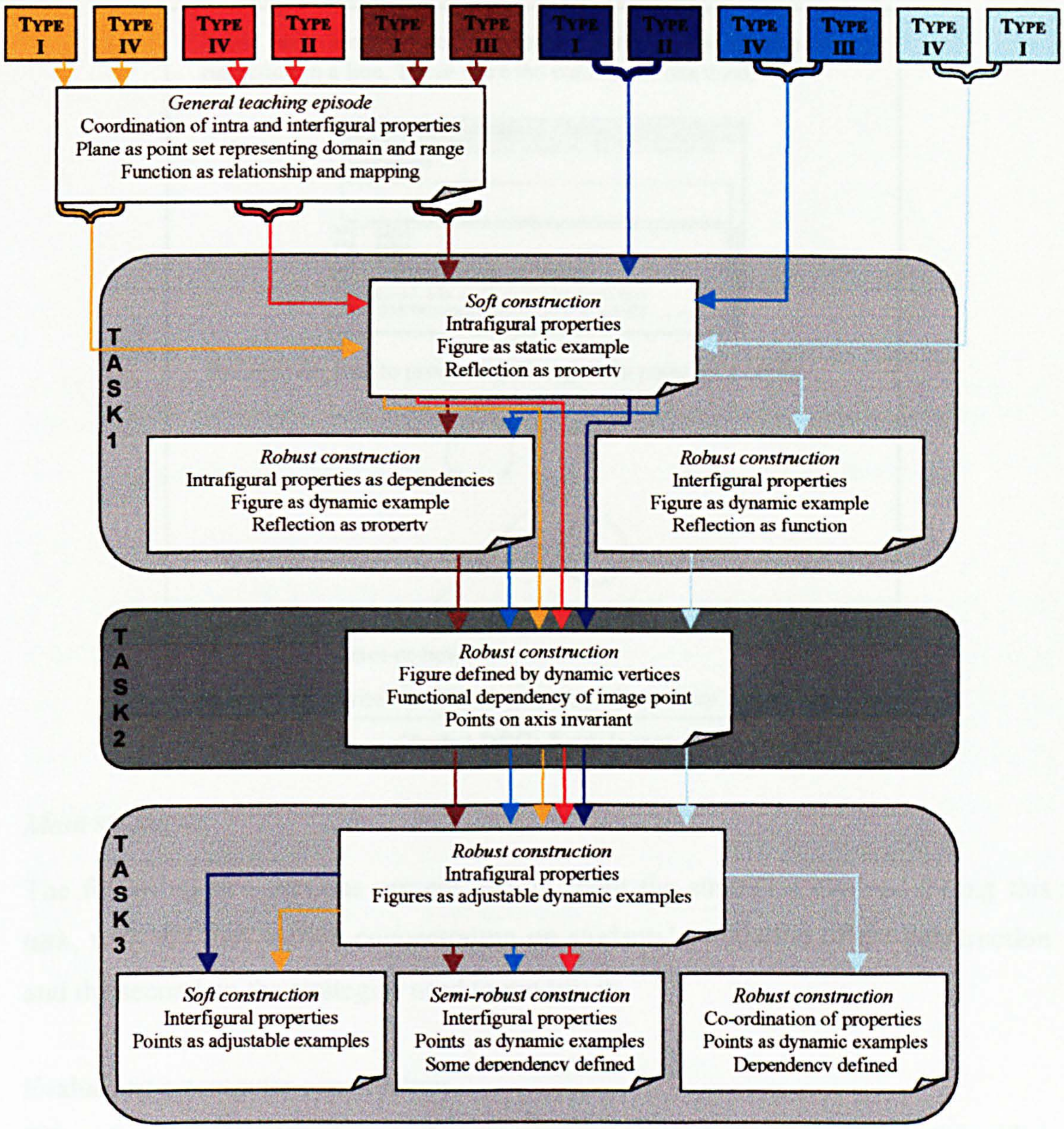
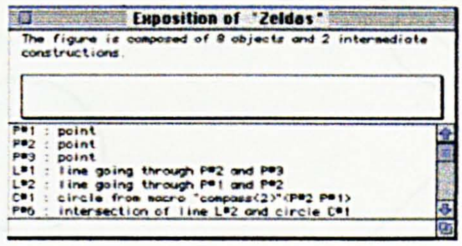


Figure 6.14: Student trajectories through the DEG learning systems to the end of third microworld task

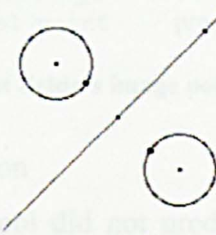
6.2.1.4 Task 4: Interpreting a formalisation of a transformation

In the fourth task, students had a new opportunity to make connections between symbol and visual representations of a transformation as they interpreted and evaluated the construction of a fictitious student, Zelda (Task 4 DEG).

Zelda built her own tool to construct the image of a point by reflection in a line. These were the commands she used:



She used her tool to produce the image of a point on a circle.



Check whether Zelda's tool produces the correct image-point always, sometimes or never.

If necessary, write a modified version of the tool for Zelda.

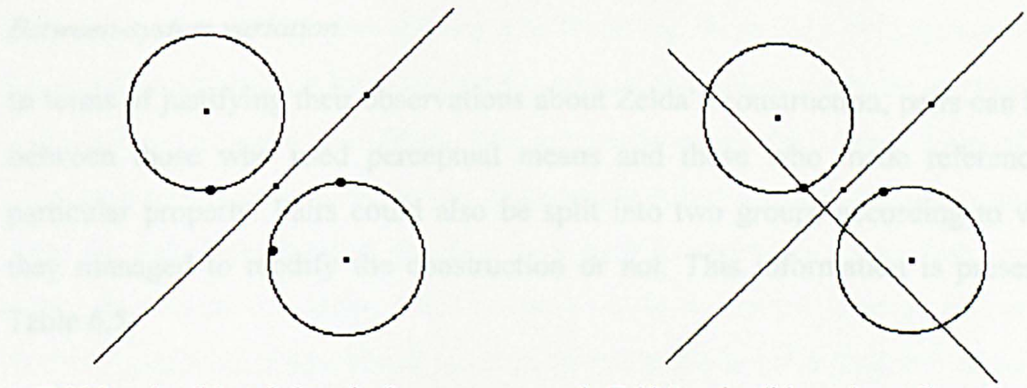
Task 4 DEG: Zelda's tool

Main strategies

The following two sections present details about the strategies evolved during this task, with the first section concentrating on students' evaluation of the construction and the second on the strategies used to modify it.

Evaluating a computer construction

When the file related to this challenge was opened, students were confronted with a figure that appeared to be symmetrical about a line. It was when the pre-image point on a circle was dragged that reflective symmetry was destroyed. Before the point was dragged, all the students correctly recognised the reflective symmetry property of the initial figure displayed on screen. When the point was dragged, however, not all were sure if its behaviour corresponded to reflection until they added further construction items – either a point constructed using the symmetrical point tool (Figure 6.15a) or a perpendicular line passing through the pre-image point (Figure 6.15b).



a: Zelda's point did not behave in the same ways as a point produced by the `symmetrical point` tool

b: Zelda's point did not always lie on line though pre-image point and perpendicular to axis of reflection

Figure 6.15: Illustrating that Zelda's image point was not the reflective image

Modifying a computer construction

Having identified that Zelda's tool did not produce a reflective image point, most students tried to reconstruct what she had actually done, reproducing her method as shown in the symbolic text on paper step-by-step. In three cases, this led to the definition of a robust construction of the reflection transformation using equal perpendicular distances (Figure 6.16). That is, students replaced the line drawn by Zelda with a perpendicular one and retained her `compass(2)` construction. Students were helped to define these constructions as macros that were added to the construction menu.

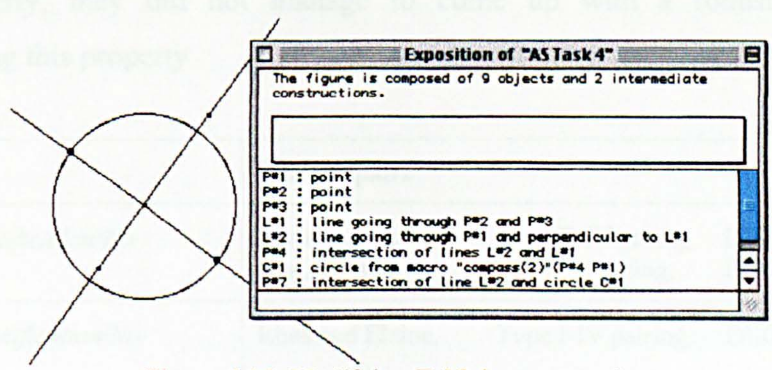


Figure 6.16: Modifying Zelda's construction

The rest of the students, while able to indicate where the true image point should lie, did not manage to modify Zelda's tool or build a robust construction of their own.

Between-system variation

In terms of justifying their observations about Zelda's construction, pairs can be split between those who used perceptual means and those who made reference to a particular property. Pairs could also be split into two groups according to whether they managed to modify the construction or not. This information is presented in Table 6.5.

Both pairings which included a type II student used perceptual means to evaluate the construction: one pair immediately rejecting the construction as a result of visual feedback from dragging and the other using the visual feedback from the `symmetrical point` tool. As they had done in other activities, both these pairs found ways to use the DEG tools that avoided formalising any properties. Neither went on to produce a correct modification of Zelda's tool. The third pair who used the `symmetrical point` tool to illustrate the problems with Zelda's image was the Type III-IV pairing from the FO system, who did go on to formalise equal perpendicular distances as they modified the incorrect tool.

One pair made use of a soft-constructed perpendicular line to show the problem with Zelda's construction. Unlike the two pairs who used the DEG tool to construct the same property, they did not manage to come up with a robust construction incorporating this property.

Strategy	**Student pairs**		
Perceptual justification/No construction	Suzie and Christie, Sita and Anju,	Type II-IV paring, Type I-II paring,	DEG-FI DEG-FO
Theoretical justification/No construction	Rhea and Elaine,	Type I-IV pairing,	DEG-FI
Perceptual justification/Robust construction	Seema and Kylie,	Type III-IV pairing,	DEG-FO
Theoretical justification/Robust construction	Anita and Sharmila, Rebekka and Maia,	Type I-III pairing, Type I-IV pairing,	DEG-FI DEG-FO

Table 6.5: Distribution of student pairs according to strategies used for fourth task

The trajectories through the systems thus far of the three pairs who produced robust constructions seemed to have afforded a sense of the difference between constructing and drawing which was coupled with increasing confidence about the geometrical relationship between pre-image and image point pair. Of the three, Seema and Kylie were the most impressed to see their own method appear as a new item, `ourway`, under the construction menu. Their reaction can be compared, in particular, to that of Rebekka and Maia who seemed to be more interested in collecting new construction methods than in naming them.

Between-system variation

There were no differences in the interactions of the pairs that could be interpreted in terms of the different instructional approaches and once again the interventions in both systems tended to involve the same issues, tool-use and validation. In the case of tool-use, macro-making required input by the researcher and, in connection to validation, not all students offered reasons to explain their evaluations of Zelda's tool unless they were requested.

Constraints and affordances

This task had been designed to illustrate that the reflection transformation is only one of a group of isometries in which distances between points are the same in the image and the pre-image. In fact, there was some initial confusion about whether it should be classified as a reflection or not. Not all students were able to correctly assess when reflection is satisfied visually, but those who could not do this, did by this stage know about a DEG tool they could use to help verify the construction.

The interactions of all student pairs suggested that their DEG interactions had encouraged them to look beyond specific drawings and focus on the behaviour of dynamic objects. Yet, some still interacted with the system in ways that avoided formalisation of particular geometrical properties – even when they seemed to be aware of the existence of them.

By engaging with what was wrong as well as what was correct about a given construction, some students managed to also identify what was lacking in their own.

The symbolic text had an important role in this. It signalled a way of constructing equal distances that enabled some pairs to complete their own reflection function. Rather than the researcher having to intervene to encourage the use of the `compass` tool, its inclusion in Zelda's construction provided a more neutral presentation, allowing students to make their own connection to its function.

Figure 6.17 presents the trajectories through the DEG systems to this stage. It shows that half the students had expressed reflection as a function in which the dependency of the image point on the pre-image point and axis was defined in terms a specific set of interfigural relationships. No evolutions in the thinking of the rest of the students from the third to the fourth task were evident in their interactions, nor did they build new computer constructions.

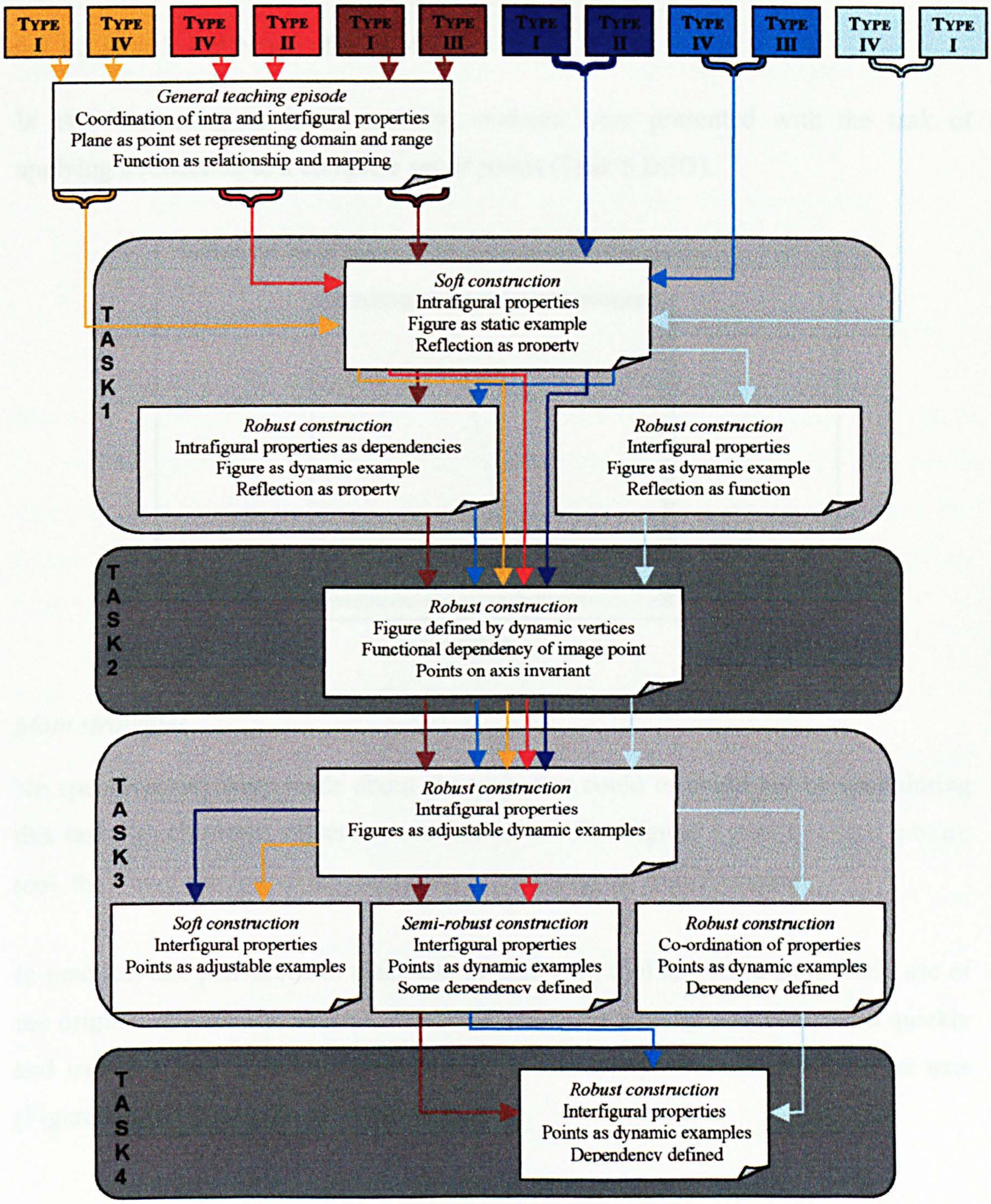


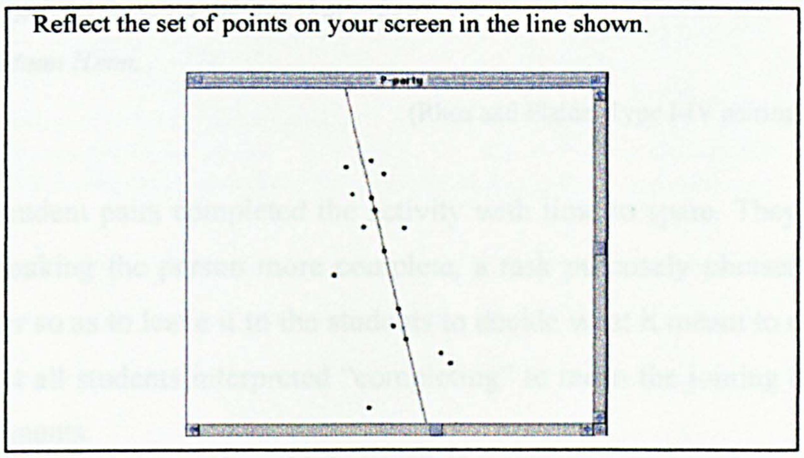
Figure 6.17: Student trajectories through the DEG learning systems to the end of fourth microworld task

Microworld evolutions

By the end of this challenge, the microworlds of three pairs of students now contained least one reflection tool of their own definition, which they had the choice of using during the final task.

6.2.1.5 Task 5: Reflection of a set of points

In the final computer-based activity, students were presented with the task of applying a reflection to a complete set of points (Task 5 DEG).



Task 5 DEG: P-party task

Main strategies

No specifications were made about the tools that could or could not be used during this task, which meant students were free to use the original `symmetrical point` tool, their own version, or any other way of enacting the transformation.

In practice, one pair chose to make use of their own tool and three pairs made use of the original microworld version. For these pairs, the activity was completed quickly and smoothly as the tool was applied to each of the points both sides of the axis (Figure 6.18).

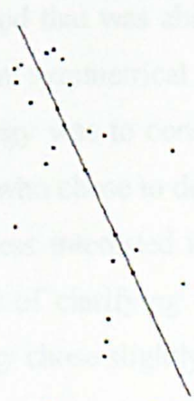


Figure 6.18: Pre image and image points in the DEG microworld

The five points located on the axis entered the activities of only one of these pairs of any of these three pairs, who decided it was not necessary to construct their images:

Rhe: *What about these ones, then? Do we need to do them.*

Ela: *I don't think so, I don't think you do them if they are on the mirror line.*

Rhe: *They just stay on the line don't they.*

Ela: *Mmm Hmm.*

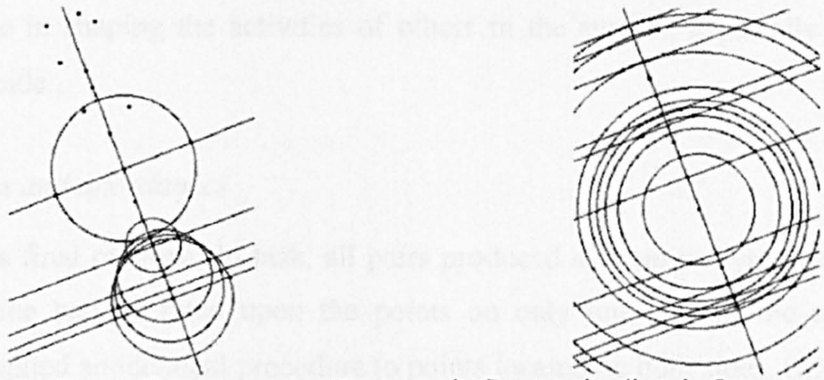
(Rhea and Elaine, Type I-IV pairing, DEG-FI)

These four student pairs completed the activity with time to spare. They were given the task of making the person more complete, a task purposely phrased in a rather vague manner so as to leave it to the students to decide what it meant to complete the figure. In fact all students interpreted "completing" to mean the joining of the points with line segments.

Between-pair variation

For the four pairs who had used a tool or macro to reflect the pre-image, visual aspects of the microworld appeared to be paramount. The tool took care of the geometrical relationships whether the built-in tool or the pairs' own macro was used and the only observable difference was the pleasure of the students who used something of their own making (evident in references to "*doing it ourway*"). Once again the type II students were among those who chose a strategy which avoided the making explicit of particular properties.

There were two variations to the main strategy (summarised in Table 6.6). One was to soft-construct each point – a method that was abandoned when the students using it were told of the much more efficient symmetrical point method by another pair in the learning system. The second strategy was to construct robustly each image point in turn. It could be that the two pairs who chose to do this, despite having access to their own macros for reflection, were less interested in the final visual product and had adopted the slightly different goal of clarifying that the same construction process could be applied to each point. They chose slightly different constructions methods as shown in Figure 6.19.



a: Construction lines in figure constructed by Anita and Sharmila

b: Construction lines in figure constructed by Maia and Rebekka

Figure 6.19: Constructions built using `perpendicular line` and `compass (2)` tools

These two last pairs both gave some attention to the points on the axis, but, again, both pairs decided, since they would remain the same, there was no need to construct their images.

Strategy	**Student pairs**
Soft construction (abandoned) Equal perpendicular distances constructed by eye	Rhea and Elaine, Type I-IV pairing, DEG-FI
Robust construction (`symmetrical point` tool)	Suzie and Christie, Type II-IV paring, DEG-FI Sita and Anju, Type I-II paring, DEG-FO Rhea and Elaine, Type I-IV pairing, DEG-FI
Robust construction (pair-defined `ourway` macro)	Seema and Kylie, Type III-IV pairing, DEG-FO
Robust construction (`perpendicular line` and `compass(2)` tools)	Anita and Sharmila, Type I-III pairing, DEG-FI Rebekka and Maia, Type I-IV pairing, DEG-FO

Table 6.6: Distribution of student pairs according to strategies used for the fifth task

Between-system variation

There was little to distinguish between the strategies used in the FI as compared to the FO system, with little need for the researcher to intervene in either. However, in the FI system, a student from one pair influenced the activities of another when she told them the strategy she and her partner were using. Both pairs seemed to be intent on completing the reflection as quickly as possible and the symmetrical point tool provided an efficient means to do so. Such a direct intervention rather contravened the intervention strategy planned by the researcher, but indicates how students too can have a role in shaping the activities of others in the system, especially when their goals coincide.

Constraints and affordances

During this final microworld task, all pairs produced a dynamic symmetrical screen design. None had operated upon the points on only one side of the axis and all students applied an identical procedure to points located on both sides. One sub-set of points that did receive a different treatment was those located along the axis. None of the students constructed the image point of these, with overriding view being that these points stay the same under reflection, there is no need to include them in the reflection process. It could be argued that this brings one back to the intrafigurallly associated aim of constructing visual representations of symmetrical figures.

Another possibility is that students were still thinking in terms of a functional relationships between pre-image and image point pairs, but that in the view of function as a relationship between two point sets, points that are invariant may be considered apart because it is not obvious that they can be described in terms of the same relationship that connects the other point pairs. Or perhaps students don't think to produce image point because invariant points are treated as exempt – why transform points to which "*nothing happens*"?

The actions of the pairs asked to complete the stick-person suggests that, as during the other DEG tasks, students were constrained to treat figures as heterogeneous collections of line segment and points. Indeed this is not surprising given the relationship between points and other objects in the DEG microworld: for a point to be considered a member of an object, that object must be present on screen before the point. This may not be particularly conducive to developing views of lines, lines segments and circles as point-sets[2]. Furthermore, the notion of the plane the DEG microworld seems most likely to engender is the space in which this moveable set of

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[2] It could be argued this does not apply to the locus tool, although during this task sequence, students appeared to see the output of this tool as a trace rather than a set of points.

different objects is contained. There was no evidence that students' activities with the DEG tools afforded any connection to the theoretical notion of plane as a set of points located in a two dimensional system, even among the students in the FI systems to whom this idea had been stressed during the general teaching episode and the introduction of microworld tools.

The diagram in Figure 6.20 presents student trajectories through the learning systems until the end of this task. It shows that the last microworld task had not involved all students in explicit attention to any properties of reflection, although all had built robust constructions treating reflection as a function. Some limits in their thinking about the functional aspects were suggested in their constructions: they appeared not to see points on the axis as part of the domain of the reflection function.

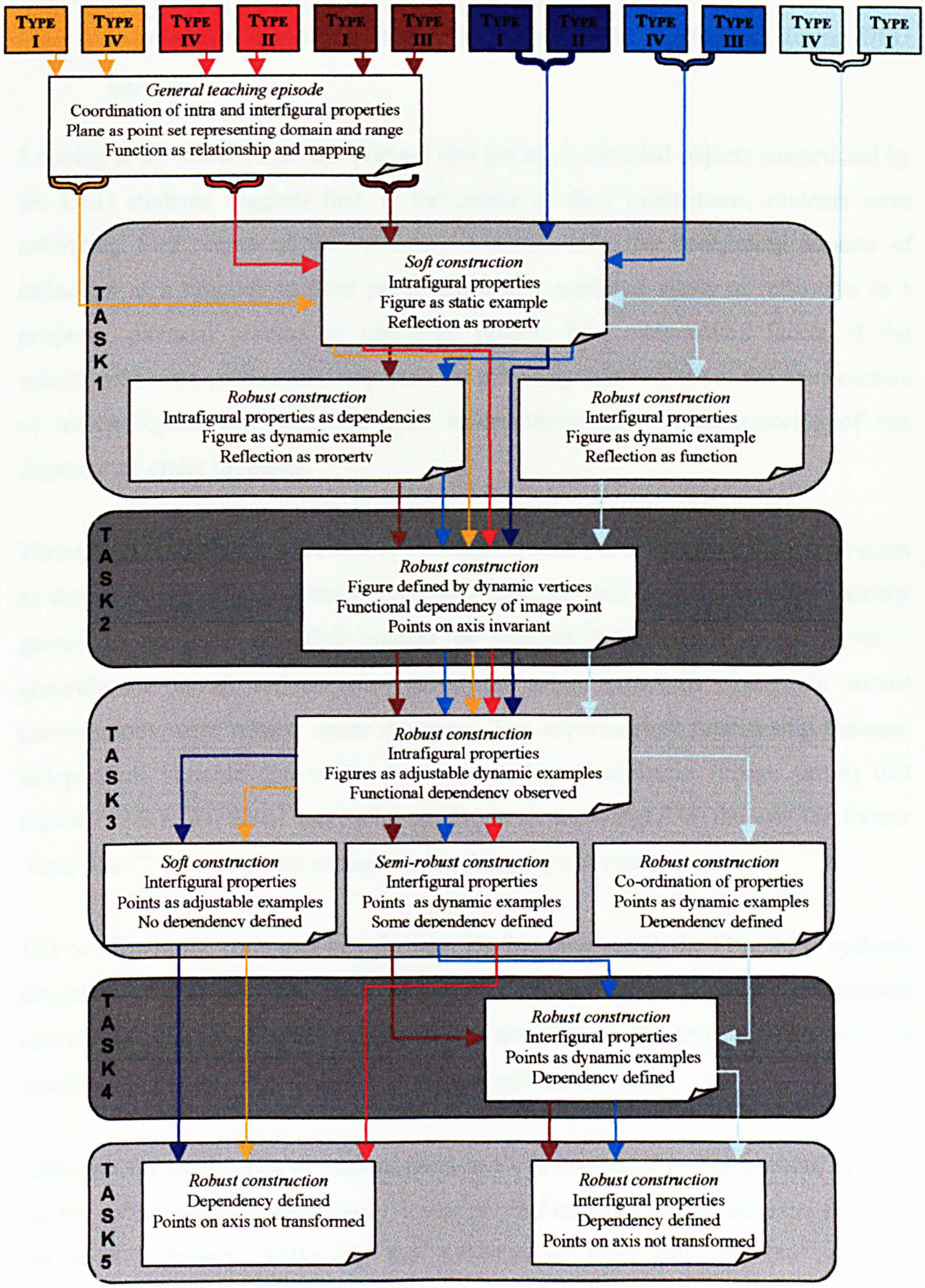


Figure 6.20: Student trajectories through the DEG learning systems to the end of fifth microworld task

6.2.1.6 Summary of evolutions in students' thinking about reflection during DEG interactions

Looking at the knowledge incorporated into the computational objects constructed by the DEG students suggests that, in the course of their interactions, students were extending their views of the reflection transformation by connecting aspects of reflection as a function to their paper-and-pencil mediated views of reflection as a property, physical process or perceived object. Two inter-related facets of the microworld were particularly important in affording this evolution: the construction of screen objects that depended upon other objects and the experiencing of this dependence under dragging.

Through the exploration of robust constructions, students came to see the image point as dependent upon the pre-image point and upon the axis. Its functional dependency gained meaning in the DEG context as students made situated abstractions – generalisable to all reflections – about the ways in which objects in robust constructions were related under dragging. The asymmetrical relationship between independent variable (pre-image turtle) and dependent turtle (image turtle) that characterises a functional approach was also made meaningful by the way the former '*controlled*'[3] the movement of the latter in dragging activities.

The overlap in the computer-based strategies that emerged in the FI and FO systems suggested that it was the tools of the microworld more than the instructional approaches that provided the source of new abstractions and provided new ways of connecting to things they already knew about reflection.

Although the trajectories of student pairs in FI as compared to FO learning systems did not differ in any consistent way, it was not that case that all student pairs followed the same trajectory. Following the activities of each pair and the computer

[3] And the fact it was the students who controlled this movement may have made it more meaningful still.

constructions that resulted from them of each pair indicates that three different trajectories had evolved.

The first trajectory involved the pairs whose paths are shown in orange, red and dark-blue in Figure 6.22. These students engaged in the construction of meanings that associated reflection with aspects of function, but largely avoided the robust construction of any distance or angles properties. In this trajectory, the ready-made `symmetrical point` tool was the principal resource used to build any dependency into constructions. This tool can itself be seen as an abstraction – simultaneously a tool to construct an image point and an object which encapsulates the properties that defines it (see also, Noss & Hoyles, 1996; p.116). It allows engagement with functional aspects without requiring formalisation of specific geometrical properties. Or, to put it another way, it afforded a means of making concrete ideas about dependence and correspondence but did not necessitate connection to any particular construction tools that underpinned the behaviour of the function's output in relation to its inputs.

What characterised this trajectory was the predominance of the visual concerns over the symbolic: meanings for dependency were abstracted from the visual feedback of dragging and student-built properties were generally constructed by eye rather than formalised. Looking at the makeup of the pairs who traversed such a trajectory shows that two of the three pairs contained the visually inclined type II students. What this suggests is that students who were particularly focussed on the visual aspects of their activity could evolve ways of working with the DEG microworld that resulted in a minimal use of construction tools.

The interactions of the two pairs in which the theoretical type III students worked (marked in brown and light-blue on Figure 6.20) evolved into a second trajectory. The trajectory most closely matched that envisaged in the design of the task sequence and was characterised by the gradual development of robust constructions. Students began to define dependencies in the first task, whilst still engaging in intrafigural analyses and, as they became increasingly aware of the difference between creation and construction, went on to make sense of various construction tools, eventually building their own robust reflection construction. A particular feature of this trajectory was that it was when students had access to symbolic as well as visual representations of robust constructions that they were best able to reconstruct them for themselves. An important difference distinguishing the activity of students who evolved this trajectory as compared to the previous one was that while visual feedback acted as an important concretising resource in the first trajectory, symbolic information also played a concretion role for the two pairs who followed the second trajectory.

The path of the pair shown in turquoise in Figure 6.20 shows a third trajectory through the task sequence, which can be characterised by the incorporation of interfigural analyses from the students' first encounter with robust constructions. Their interactions with these constructions appeared to have opened a new window onto reflection for the students and they connected the visual feedback with construction tools without recourse to symbolic descriptions. The task sequence had been designed with Piaget and Garcia's view of intra and inter as successive stages in an epistemological hierarchy, but this pair seemed to use interfigural considerations to inform intrafigural analyses rather than the other way round. In the DEG context at least, the direction between intra and inter is better considered as bidirectional rather than unidirectional.

The emergence of three distinct trajectories through the set of tasks shows how interactions of all student pairs were mediated by the external resources available in form of microworld tools and task, but that students' relationships with these resources were not uniform – they were also mediated by the internal resources of each participating student. Their specific trajectories were shaped by students' own preferences for particular forms of representation and their own interpretations of the aims of the task. This underlies how the play paradox enters into considerations of microworld-mediated mathematics learning – in these systems it appeared to have a stronger impact than planned instructional approaches. The tools of the DEG kernel were designed to both constrain and respect diversity in student strategy and this meant giving students a certain freedom in expression. The tools did constrain activity in such a way that all students confronted issues related to function. At the same time, they also supported a diversity that resulted in the pursuit by some students of goals that did not always match those intended in the task design. There were, however, some signs of a convergence in goals across all pairs as the learning systems evolved: some students appropriated the idea of building their own external resource for reflection early in the sequence; others were only beginning to want to develop constructions that did not mess up by the end.

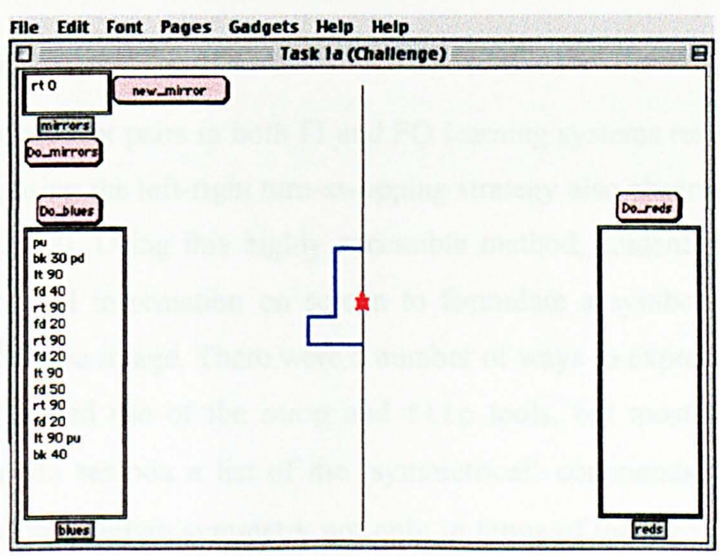
6.2.2 Microworld interactions with the MTG learning systems

Having considered the microworld interactions in the two DEG learning systems, attention now turns to students' interactions around each of the five MTG tasks.

6.2.2.1 Task 1: Completing figures with reflective symmetry in the MTG systems

Part 1 of this task is shown in MTG 1a. Students were to use reflection to complete a design given its first half and the Logo commands that had produced this half.

Complete the design on your computer screen using the red turtle to trace the reflected shape.



Check that your method works when the angle of the mirror turtle is changed.

Task 1a MTG: Part 1, Introductory figure

Part 2 presented students with a variable procedure that defined half a stick-person, which could be drawn with its arms and legs orientated at different angles to its body (Task 1b MTG). Students' task was to complete the stick-person by reflection.

Use the Logo procedure for half a person to help in the construction of a complete symmetrical stick-person.


```
to per :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

Check your procedure works by drawing the symmetrical person in different ways.

Task 1b MTG: Part 2, A symmetrical person

Main strategies

The majority of student pairs in both FI and FO learning systems resolved both parts of the first task using the left-right turn-swopping strategy also observed in the design phase (see *§5.1.2.2*). Using this highly accessible method, students coordinated the symbolic and visual information on screen to formulate a symbolic expression to produce the reflective image. There were a number of ways to express the reversal of turns. These included use of the `swop` and `flip` tools, but most common was to record in the `reds` textbox a list of the 'symmetrical' commands. On screen, this method resulted in a certain symmetry not only in terms of the visual traces but also in the symbolic language contained in the textboxes (see Figure 6.21).

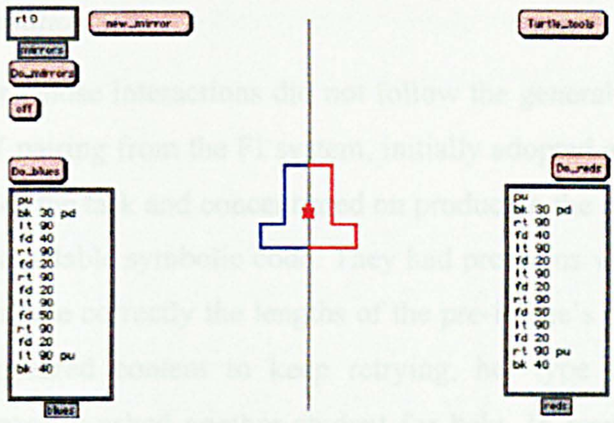


Figure 6.21: Symmetry in the visual and the symbolic

The turn-swopping relationship encapsulates the properties of a symmetrical figure in a form that students find easy to concretise – it connects not only with the world of the turtles but also with what they know about reflection in mirrors where left and right are also reversed.

In the second part of the task, the same relationship was used to aid students in the construction of a variable person procedure. The majority defined a second variable procedure that was a copy of the first except that the orientation of each angle command was swopped. Figure 6.22 presents an example, showing the procedures used by one student pair. The example is representative in that students gave the procedures 'half-names' and referred to the figure as a whole, in this case calling it "*Simon*" – changing the name of the given procedure `per` but following the implied practice of seeing each procedure as half of a whole symmetrical figure.

```
to si :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

```
si 45 20
mon 45 20
```

```
to mon :sh :hip
pu
fd 80 pd
rt 90 fd 10 rt 90
fd 20 rt 90 fd 10
lt 90
fd 10
lt :sh
fd 30 bk 30
rt :sh
fd 50
lt :hip
fd 50 bk 50
rt :hip
rt 180
end
```

Figure 6.22: Example of procedures used to construct symmetrical stick-person

Between-system variation

There was one pair whose interactions did not follow the general pattern. Alissa and Helen, a type II-IV pairing from the FI system, initially adopted a perceptually based approach to Part 1 of the task and concentrated on producing the correct trace without making use of the available symbolic code. They had problems with their image as it was difficult to estimate correctly the lengths of the pre-image's segments. While the type II student appeared content to keep retrying, her type IV partner became frustrated and eventually asked another student for help. In response, she was told about the turn-swopping strategy and she persuaded her partner to use this as they tackled the second part of the task. They had spent so much time on the first part of the task that they did not manage to complete their own variable procedure and, at the end of the task, asked for a copy from another pair.

Between-system variation

The turn-swopping approach emerged similarly in the two MTG systems as well in all the iterations of the design phase and in previous research projects. This would suggest that it was the MTG tools that encouraged students to abstract the relationship rather than it being associated with the global structuring of one or other instructional approach.

In terms of local structuring, researcher interventions were made at similar points in the two systems and especially concerned technical issues related to defining and running variable procedures. In the FI system, a student intervened in the work of another pair as described above. This intervention was very direct and did not follow the planned intervention approach of the researcher, although the pair who were 'given' a strategy did have some success in concretising it for themselves.

Constraints and affordances

Students' interactions during both parts of this task were directed entirely towards intrafigural analyses, with the aim being the production of figures with reflective symmetry. As such their activities can be characterised as the concretion of knowledge used in paper-and-pencil activities within the MTG context. Students had to express this knowledge through the symbolic code of the MTG microworld and, in doing so, angle as well as distance properties were incorporated into the constructions of all the pairs.

The MTG-situated turn-swopping abstraction resulted from the synthesis of the visual and the symbolic into a simple rule; perhaps this was its strength and its wide appeal. Students inclined to focus on visual aspects could see that the symbolic code had the desired effects. Similarly, for those who tended to emphasis theoretical aspects, the symbolic code provided a new way of expressing the properties of figures with reflective symmetry while also producing a visual representation.

In terms of thinking about figures, the need to communicate with turtles constrained students to treat figures as turtle paths, which could not be operated upon once drawn on screen but might be encoded into procedures. In this way, a figure drawn on screen resembled one drawn on paper – it could not be directly manipulated. Once encoded into a variable procedure, however, its status changed and it became a representative of a set of figures whose generality resided in the symbolic data – a set of stick-people with arms and legs that could be inclined at different angles to the body.

Turning figures into procedures in this task also allowed students to bring some sense of their own identity to the learning system. Giving computational objects names they cared about was another way in which students made connections to them, with the names indicative of affective aspects of the meanings students were constructing for the resources they built.

In summary, during this task, students had evolved a new way of expressing the intrafigural properties of symmetrical figures, in turns of the turns and distances in turtle paths. Students' interactions had not, however, suggested they were thinking about any functional aspects of reflection. Figure 6.23 summarises the way knowledge about reflection evolved into the computational objects constructed during the first task and shows the paths of the students through the learning systems thus far.

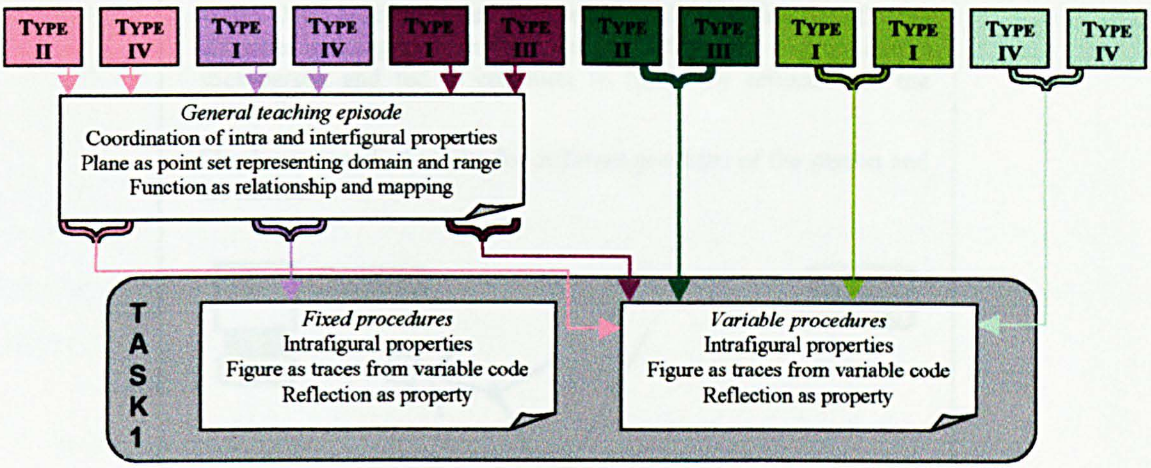


Figure 6.23: Student trajectories through the MTG learning systems to the end of first microworld task

Microworld evolutions

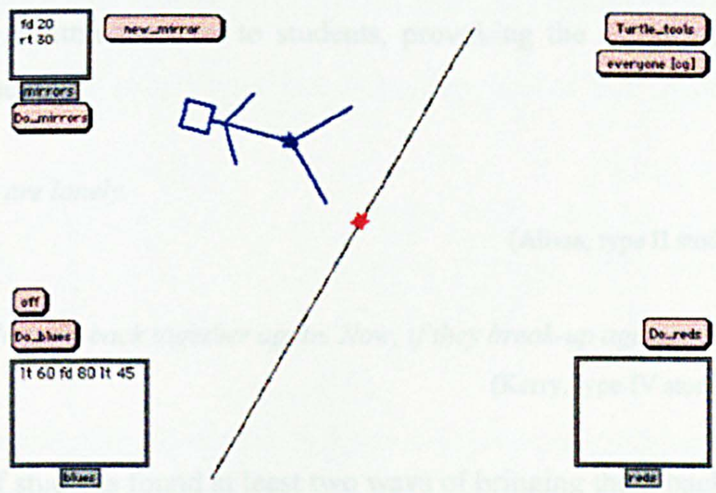
The extensions to the MTG microworlds were uniform across pairs in both systems: all now included procedures for making the variable stick-person that would be used in the next task.

6.2.1.2 Task 2: Reflection as a relationship between image and pre-image

This task (Task 2 MTG) was intended to shift students' attention from intrafigural to interfigural relationships. Rather than producing symmetrical figures, students were to produce reflections of the symmetrical figures of the last task and then further explore the relationships between pre-image, axis and images by making symmetrical turtles coincide on the axis.

Write ONE procedure that can be used to send blue to different distances and angles from the mirror line. Use blue to produce the stick-person and red to construct its image by reflection in the mirror line.

Check your method works for different positions of the person and the mirror.



Find different ways to reunite red and blue on the mirror line.

Task 2 MTG: Reflecting people

Main Strategies

The problem of positioning a stick-person at some distance from the axis and then constructing its images was solved in essentially the same way by all six pairs – swopping the orientation of turns in the path of the blue turtle to determine the path of the red, as illustrated in the following exchange.

Lau: *OK so now, we have to use the red to do the image.*

Can: *Just change the lefts and rights again, how did we send the blue?*

Lau: *It was...lt, left 60, forward 80 left 45 then PD for freddie.*

Can: *Putting it round the other way.*

```
<rt 60 fd 80 rt 45 pd die 90 45 fred 90 45>
```

Lau: *Yes!! ... Now, a different distance and angle.*

(Candy and Laurel, type I-I pairing, MTG-FO)

All students investigated with different locations for the stick-person and its image and with axes of different orientations. As they did so, the practice that developed was to re-use the same command sequence, maintaining its overall structure and changing only the values of the turns and moves by which the blue and red turtles were positioned into symmetrical locations.

The generation of re-usable command sequences also occurred in the challenge of reuniting the blue and red turtles. As had been the case during the design phase, this challenge was an attractive one to students, provoking the use of anthropomorphic descriptions, such as:

Perhaps they are lonely.

(Alissa, type II student, MTG-FI)

We've brought them back together again. Now, if they break-up again...

(Kerry, type IV student, MTG-FO)

The majority of students found at least two ways of bringing them back together, with three strategies evolving:

- Working backwards through the commands used to position blue and red (Figure 6.24).

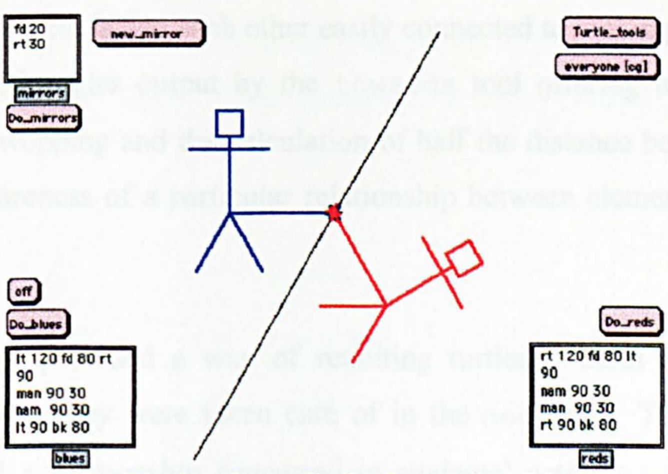
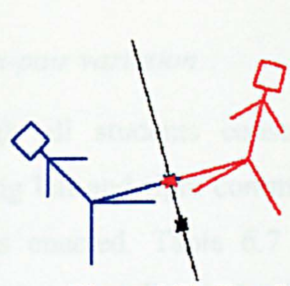


Figure 6.24: Moving the turtles back to their original location

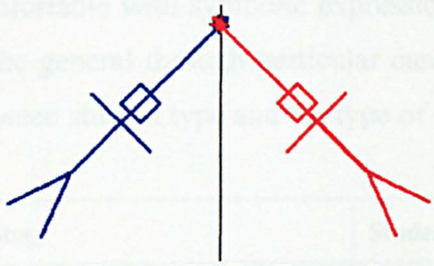
- Moving the turtles to the midpoint between them by turning them to face each other and then asking them to go forward half of the distance between them (Figure 6.25).



```
blue, show towards "red
                                    {rt 120}
blue, rt 120
red, lt 120
blue, show distance "red
                                    {130}
blue, fd 65
red, fd 65
```

Figure 6.25: Reuniting blue and red at the midpoint between them

- ❑ Using the meet tool to construct the intersection point of their paths (Figure 6.26).



```
blue, meet "red
red, meet "blue
show distance "red   {150}
blue, fd 150
red, fd 150
```

Figure 6.26: Sending turtles to their 'meeting' point (point of intersection)

In the first strategy, students' attention was on the turtle paths, treated as generic rather than specific lists of commands. It could be classed as a symbolically based strategy, exploiting what now had become the turn-swopping algorithm.

The second strategy appeared to be motivated by both symbolic and visual concerns, with the idea of objects facing each other easily connected to looking into mirrors, the equal but reversed angles output by the `towards` tool offering more evidence in support of turn-swopping and the calculation of half the distance between the turtles indicating an awareness of a particular relationship between elements of pre-image, image and axis.

The third strategy provided a way of reuniting turtles without first making any properties explicit – they were taken care of in the `meet` tool. This tool had been defined to model a relationship suggested in students' activity – that symmetrical turtles meet on the axis. It suggests a focus on visual concerns, although, because the tool placed a new turtle at this meeting point rather than sending the red and blue turtles to this point, it was still necessary for students to identify and execute the equal distance property.

Between-pair variation

Although all students constructed the two stick-people to be symmetrical by swopping left and right commands, there was some variation across pairs in the way this was enacted. Table 6.7 shows that half the students changed the inputs to commands manually each time a turtle was moved from the axis, while half constructed variable procedures to do this. This was not indicative of different meanings for reflection, but perhaps suggests that some students were more comfortable with symbolic expressions of generality, whilst others preferred to look at the general through particular cases. There was no suggestion of any relationship between student type and the type of computer formalisation used.

Strategy	Student pairs		
Manually swopping LT and RT in direct drive commands	Lizzie and Aimee, Candy and Laurel Prija and Jodie,	type I-IV paring, type I-I pairing type II-III pairing	MTG-FI MTG-FO MTG-FO
Variable procedures	Hadley and Lorna, Alissa and Helen Kerry and Sophy,	type I-III pairing, type II-IV pairing type IV-IV pairing,	MTG-FI MTG-FI MTG-FO

Table 6.7: Distribution of student pairs according to strategies used to position blue and red turtles in symmetrical positions

Table 6.8 shows how students were spread among the three strategies used to reunite the red and blue turtles. Again, no relationships between student type and strategy were found.

Strategy	Student pairs		
Reuniting red and blue on the mirror turtle (Symbolic aspects emphasised)	Hadley and Lorna, Prija and Jodie, Kerry and Sophy,	type I-III pairing, type II-III pairing type IV-IV pairing,	MTG-FI MTG-FO MTG-FO
Reuniting red and blue on their 'midpoint' (Symbolic/visual mix)	Hadley and Lorna, Alissa and Helen Candy and Laurel,	type I-III pairing, type II-IV pairing type I-I pairing,	MTG-FI MTG-FI MTG-FO
Reuniting red and blue on their meeting point (Symbolic aspects emphasised)	Hadley and Lorna, Kerry and Sophy,	type I-III pairing, type IV-IV pairing,	MTG-FI MTG-FO

Table 6.8: Distribution of student pairs according to strategies used to reunite blue and red turtles at a point on the axis

There was one pair, Lizzie and Aimee, a type I-IV pairing from the FI system, who do not appear in Table 6.8. They ignored the last part of the task and made up a different activity of their own, which involved finding ways of producing the two stick-people so that the ends of their arms met on the axis – they wanted the two stick-people to appear as if they were holding hands (Figure 6.27).

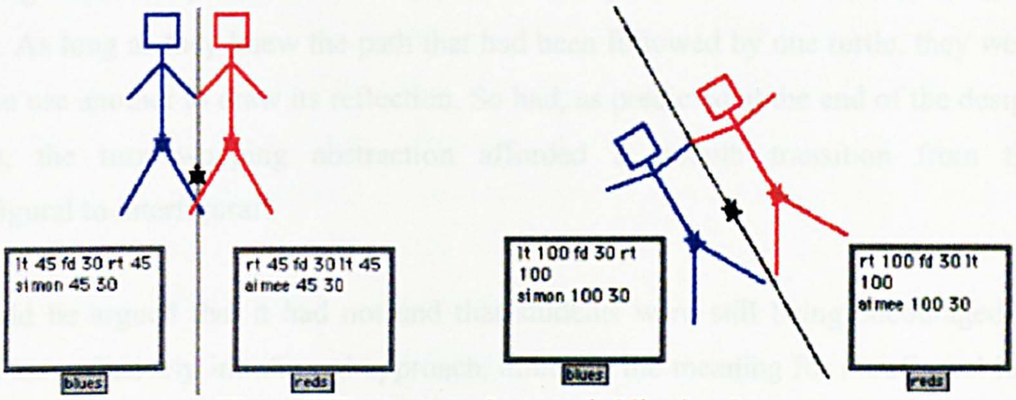


Figure 6.27: `simon` and `aimee` holding hands

Between-system variation

Although there were some differences in the work of students following the FI approach as compared to the FO pairs, these differences did not seem to be connected to the global structuring of the learning system. Instead, the variations related to how the students themselves engaged with the tasks – the more involved students became with the activities, the more they seemed to add particular goals of their own. As they pursued these goals, their relationships with computational objects they were constructing seemed to lead to the shaping of the instructional approach rather than being shaped by it.

For example, in the FI system, the students asked for more help in writing variable procedures than those in the FO system, as two pairs expressed rather particular motivations for wanting to use variables. In one case, students in the pair who copied a variable procedure in the previous activity were determined to build one of their own. In the other, students had wanted to combine two variable procedures into superprocedures.

Although there were some between-system differences on the use of variables, there were similar patterns of use of the communication tools in both systems, with all needing help with syntactic issues, especially in connection with the tools used for obtaining information about angle and distance measures.

Constraints and affordances

By the end of this task, all student pairs in both systems had developed a way of thinking about the production of a reflection firmly connected to the idea of turtle paths. As long as they knew the path that had been followed by one turtle, they were able to use another to draw its reflection. So had, as predicted at the end of the design phase, the turn-swopping abstraction afforded a smooth transition from the intrafigural to interfigural?

It could be argued that it had not and that students were still being encouraged to adopt an exclusively intrafigural approach, although the meaning for intrafigural had changed a little in the MTG context. Instead of figures, students were now operating with turtle paths. So, the pre-image that students had to reflect included the part of the screen design produced as the turtle moved into position (with its pen up) as well as the trace left when it moved with its pen down. Because of the way the task had been structured with the three turtles – `blue` who drew the pre-image, `red` who produced the image and `mirror` who defined the axis – initially sharing the same turtle-state, there was no need for students to directly define the relationship between pre-image turtle and image turtle – rather they were both defined in terms of the common point from which their journeys begun. And, as it happened, this starting point also defined the axis.

With turtle paths as the object of students' attention, their activities were still characterised as the making of comparisons between their internal properties, which, as described in *§5.1.1.2*, Piaget and Garcia would classify as an intrafigural analysis. Thinking of a pre-image as the path of one turtle and its image as the path of a second connects with those views expressed by students in their paper-and-pencil work in which the application of reflection is not limited to figures alone, but to everything visible in the pre-image (see *§6.1.1*). In their MTG interactions, students did seem to be extending their knowledge of reflection: turtle paths confine the reflection to a two-dimensional space, without confining it to one half of the plane; turtle paths also necessitated making explicit in symbolic code the distance and angle property. This led all students to come up with visually correct images. But the tools had made it possible for students to construct reflective images reliably without adopting an interfigural perspective and without considering its functional aspects.

It was when it came to reuniting the turtles that some of the students shifted attention from intra to interfigural relationships. Those who did more than send the turtles back along the paths used to position them, had had to think about the position of the turtles within a surrounding space – abstracting a relationship between pre-image and image turtles and another object, the axis, external to them. As they did so, students seemed to be treating particular cases as generic examples – the turtles became generic objects that could be positioned anywhere on the screen and the paths that connected them were also treated as generic, as students showed an awareness that the same relationships could be used to reunite them regardless of their particular state on screen.

Students' trajectories through the MTG learning systems to the end of second task are summarised in Figure 6.28. It shows that they built re-usable command sequences in both parts of the task, but the focus changed from intrafigural analysis of turtle paths, to interfigural relationships between turtles.

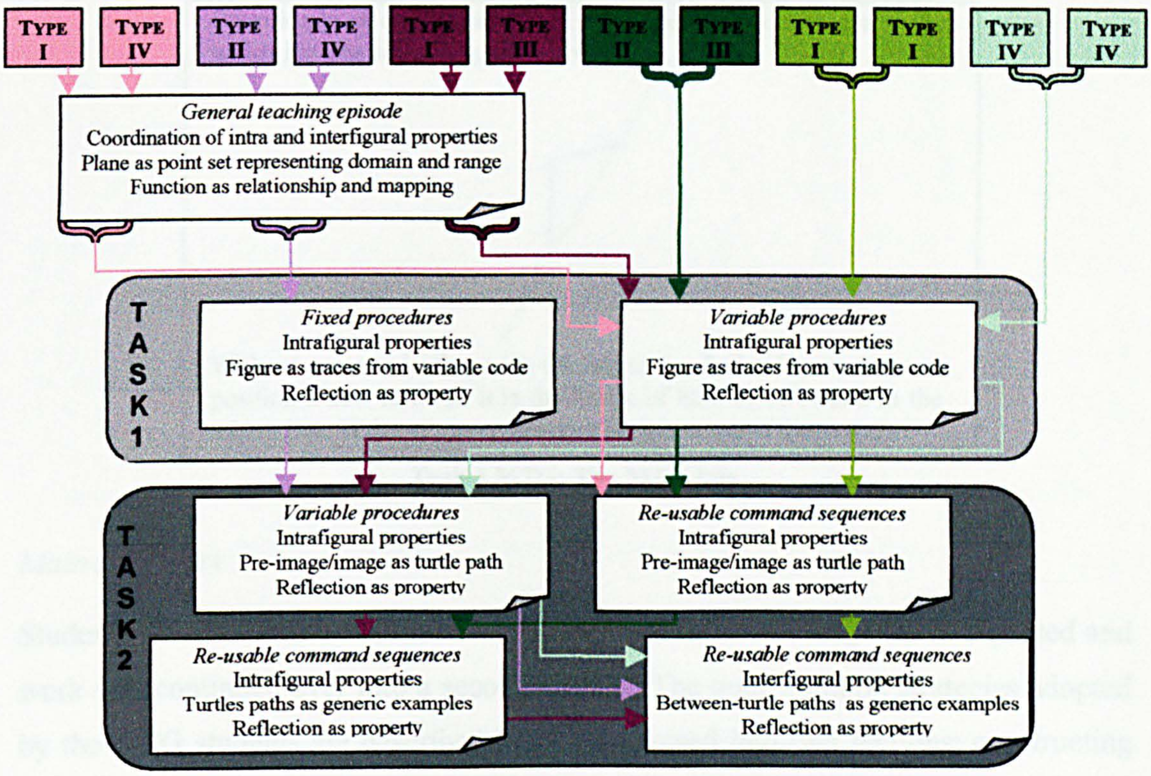


Figure 6.28: Student trajectories through the MTG learning systems to the end of second microworld task

Microworld evolutions

In terms of microworld evolutions, three of the MTG microworlds now included variable procedures that could be used in positioning turtles.

6.2.1.3 Task 3: From symmetrical paths to image turtles

The analysis of students' interactions around the second task suggested that, to embark on interfigurally orientated analyses of reflection, students needed to concentrate on relationships between turtles and that they might only be encouraged to attend to the functional relationship between pre-image turtle, axis turtle and image turtle when all three were not initially presented in the same state. To resolve this task (Task 3 MTG) students would need to take account of both these factors.

What kinds of quadrilaterals can be made by reflecting a triangle which has one side along the mirror line?

Without communicating with the red turtle, find different ways to position a new turtle so it is the image of blue by reflection in the mirror line. Write a Logo procedure based on one of these ways.

Task 3 MTG: The MTG kite

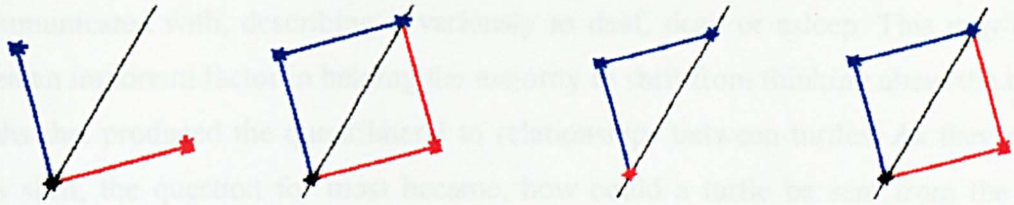
Main strategies

Students' interactions around the demands of this task took longer than expected and work was continued over into a second session. The main solution strategies adopted by the MTG students are described below, organised into two sections: constructing specific quadrilateral types; and from symmetrical quadrilateral to turtle and image.

Constructing specific quadrilateral types in the MTG systems

Constructing a quadrilateral type by reflecting a triangle in one of its sides proved a considerable challenge to the students and the majority worked only on the case of the square. In order to build a square, students had to co-ordinate what they knew about its properties and the properties of reflection from the beginning of the construction process – the first step had to involve a turn of 45° and this necessitated connecting knowledge of the right angles of squares with knowledge of equal angles either side of an axis of reflection. In practice, most students began with a different angle, but as soon as they thought the reflection of what they produced on screen, they realised that the first corner of their square would result from combining two equal turns in opposite directions and therefore the first turn must be half of 90°.

Two strategies that had also been observed in the design phase (see *§5.2.2.1*) emerged in both systems, with the *sides method* (building the sides of two symmetrical triangles simultaneously) as shown in the example presented in Figure 6.29a more common than the *triangles method* (building a triangle and then its image), an example of which is shown in Figure 6.29b.



```
blue, pd lt 45 fd 100
red, pd rt 45 fd 100
blue, rt 90
red, lt 90
blue, meet "red
remember distance "red
blue, fd :m1 bk :m1
red, fd :m1 bk :m1
```

a: A square constructed using the sides method

```
blue, pd lt 45 fd 100 rt 90
meet "mirror
remember toheading "mirror
lt 45
remember distance "blue
blue, pd fd 100 bk 100
red, pd rt 45 fd 100 lt 90
pd fd 100 bk 100
```

b: A square constructed using the triangles method

Figure 6.29: Two methods for constructing a square around an axis of reflection

As they built the square, all pairs constructed a sequence of commands that could be used as a template for the general cases of the symmetrical quadrilateral, the kite. A general intervention was made in both systems to suggest that students do this by investigating what happened if the sides and angles of the two triangles were altered, whilst the symmetry of the quadrilateral was maintained.

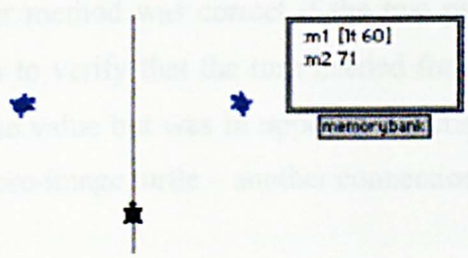
During this investigation, students not only obtained empirical data concerning the properties of the general quadrilateral, but also experienced how some properties emerge as a property of others: that is the values returned by the `remember` tool were dependent on the angles and distances the students changed manually.

From symmetrical quadrilateral to turtles and images

In the second part of the task, students were to construct a turtle with the same turtle-state as the red image turtle, without actually communicating with red. Furthermore, the method was to work regardless of the initial state of the blue turtle. By this stage in the learning system, a MTG-mediated meaning for generality had evolved, with students knowing that a valid construction method was one whose structure could be re-used with specific changes made – another example of expressing generality in action.

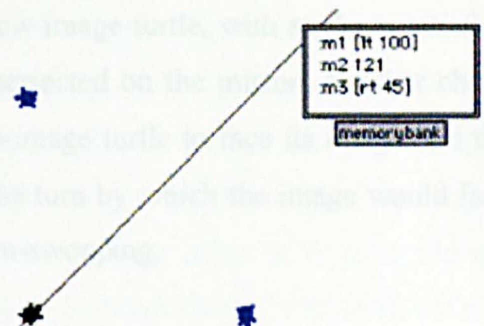
As had been the case with reuniting turtles on the previous task, many students invented their own stories to explain why the red turtle could no longer be communicated with, describing it variously as deaf, dead or asleep. This may have been an important factor in helping the majority to shift from thinking about the turtle paths that produced the quadrilateral to relationships between turtles. As they made this shift, the question for most became, how could a turtle be sent from the pre-image turtle-state to the image turtle-state?

Broadly speaking, students could again be divided between two strategies, this time distinguishable by the way the new image turtle was sent onto the axis. The *meeting method* involved starting by sending this turtle to the intersection point of the pre-image turtle and the mirror turtle (Figure 6.30a), while in the *mirror-turtle method* the new turtle was positioned on top of the mirror turtle during the construction procedure (Figure 6.30b).




```
to freda
meet "mirror
remember towards "mirror
run :m1
run :m1
remember distance "blue
bk :m2
end
```

a: An image turtle procedure involving the meeting method




```
to lhimage
hatchhere
remember towards "mirror
run :m1 rt 180
remember distance "mirror
bk :m2
remember toheading "mirror
run :m3 run :m3
fd :m2
lt 180 run :m1
end
```

b: An image turtle procedure involving the mirror-turtle method

Figure 6.30: Two methods for producing an image turtle

The starting point for the meeting method was visual, whereas the mirror-turtle method began with symbolic considerations. To be put into practice, however, some coordination of the visual and the symbolic was required for both. In the meeting method, the `meet` tool allowed students to position a turtle on the mirror without first thinking about turns and distances, but could not be completed without incorporating some formalisation of both equal turns and equal distances. The mirror-turtle method was related to the idea of undoing turtle paths, but, as the blue turtle was placed at unknown distances and angles from the mirror turtle (by dragging), students needed to think about more than the syntactic differences between turtle paths and their interpretations of feedback from the `towards` command motivated the inclusion of visual analyses alongside the symbolic.

Only half the student pairs actually turned the command sequences they used into procedures. The others worked only in direct drive, although all groups tested that the commands they used worked for different positions of the blue turtle, with those who did not write procedures changing the values in the command sequence as necessary.

As they checked their methods for different locations of the blue turtle, the red turtle was no longer available to act as an empirical check. Instead, the most common check was to ask the pre-image turtle to meet the new image turtle, with students satisfied their method was correct if the two paths intersected on the mirror. Another check was to verify that the turn needed for the pre-image turtle to face its image had the same value but was in opposite direction to the turn by which the image would face its pre-image turtle – another connection to turn-swopping.

Between-pair variation

Table 6.9 shows how the student pairs were divided between the two methods for producing the symmetrical quadrilateral. Both the pairs who chose to make use of the variables created when they used the remember tool included the systems' type III students. The FI pair, Hadley and Lorna, had consistently written variable procedures throughout the task sequence, perhaps seeking to express themselves in the most general manner possible. The FO pair, Prija and Jodie, was more erratic in its use of variables; perhaps an adaptation of the students to each other's needs, as the pairing brought together a type II with a type III student.The third pair which made use of variables in the first part of the tasks was a type I-I pairing.

Strategy	Student pairs		
Sides method			
Remembered variable used	Hadley and Lorna,	type I-III pairing,	MTG-FI
Remembered variable used	Prija and Jodie,	type II-III pairing	MTG-FO
Remembered values changed	Lizzie and Aimee,	type I-IV pairing,	MTG-FI
	Kerry and Sophy,	type IV-IV pairing,	MTG-FO
Triangles method			
Remembered values changed	Alissa and Helen	type II-IV pairing	MTG-FI
Remembered variable used	Candy and Laurel,	type I-I pairing,	MTG-FO

Table 6.9: Distribution of student pairs according to strategies used to construct symmetrical quadrilateral

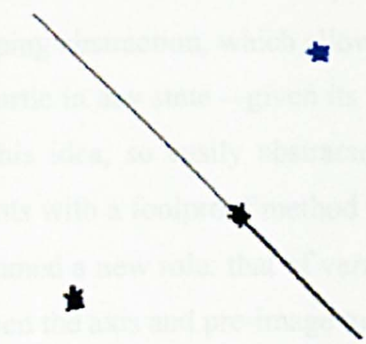
Table 6.10 shows that other student pairs went on to incorporate expressions including the remembered variables when they re-used the turtle-communication tools in their attempts to position a reflective image turtle. There were hence two different routes by which students appropriated the use of the remembered variables. Some pairs used them from the outset, creating the most general expression they could and then particularising it in practice. Others generated a number of particular examples for themselves before being motivated to make use of the variables that represented their general expression. Some did not use them at all, continuing to express generality in action throughout both challenges. In all cases, students seemed to be aware of the properties; the difference may have related to their confidence in ceding control to a computer-generated symbol, as opposed to holding on to control by manually inputting values each time an example was constructed.

Strategy	Student pairs		
Meeting method			
Procedure with variables	Alissa and Helen,	type II-IV pairing,	MTG-FI
Procedure with variables	Candy and Laurel,	type I-I pairing,	MTG-FO
Remembered values changed	Lizzie and Aimee,	type I-IV pairing,	MTG-FI
Mirror-turtle method			
Procedure with variables	Hadley and Lorna,	type I-III pairing,	MTG-FI
Remembered variable used	Prija and Jodie,	type II-III pairing,	MTG-FO
Remembered values changed	Kerry and Sophy,	type IV-IV pairing,	MTG-FO

Table 6.10: Distribution of student pairs according to strategies used to construct image turtle

In both the meeting method and the mirror-turtle method, the majority of the student pairs had begun with a new turtle who shared the turtle state of the blue (pre-image) turtle which was mapped onto its image state. One pair, Kerry and Sophy, a type IV-

IV pairing from the MTG-FO system, adopted a slightly different approach, hatching their new turtle on top of the mirror turtle rather than the blue turtle. This strategy is perhaps less suggestive of a dynamic image of function; certainly it does not actually involve mapping the initial turtle state on to the final one. The final method developed by this pair, involved creating not one but two new turtles: an intermediate turtle recreated the path of the pre-image turtle from the mirror turtle, which could be swopped to position the image turtle. The intermediate turtle was then deleted (Figure 6.31). It could be argued that this solution remains based on intrafigural analysis with the object of attention on comparing internal relationships of turtle paths rather than thinking about transforming one turtle-state to another.



```
mirror, hatchhere
show towards "blue {lt 31}
rt 31
show distance "blue {93}
fd 93
mirror, hatchhere
lt 31 fd 93
show towards "blue {lt 19}
<delete second new turtle then click on the image turtle>
rt 19
```

Figure 6.31: A variation on the mirror-turtle method

Between-system variation

There seemed to be no association between the evolution of particular strategies and instructional approach, with the same methods emerging in both FI and FO learning systems. Interventions by the researcher were required at similar points in students' interactions, the most significant being the decision to suggest that students investigate how symbolic code used to construct a square could be re-used to produce other symmetrical quadrilateral. This legitimised the practice of re-using command sequences that had evolved during previous microworld interactions – it could therefore be argued to be an intervention supportive of an FO approach to instruction. It was deemed necessary due to the time constraints imposed on the system and the desire to keep all student pairs working on the same challenges at the same time. In addition to this general intervention, student pairs often asked for help when they decided to attempt to include variables in their expressions. The help they received was of a technical nature, offered similarly in both systems.

Constraints and affordances

During this task, students' interactions suggested that they were incorporating the use of interfigural as well as intrafigural analyses and directing their attention to functional aspects of reflection. In particular, the tools and task of the MTG microworld had afforded a movement from working with reflection as property in the first part of the task to reflection as function in the second.

A number of factors were important in supporting this move. First was the turn-swopping abstraction, which allowed the students to position correctly an image turtle of a turtle in any state – given its path from an axis defining state. It has been argued that this idea, so easily abstracted during students' intrafigural analyses, provided students with a foolproof method of generating symmetrical figures. During this task, it assumed a new role: that of verifying constructions of image turtles when the paths between the axis and pre-image turtle were not known *a priori*.

The second factor was the evolving practice of re-using command sequences, which encouraged students to think about particular cases as generic examples.

Thirdly, during intrafigural analysis, the asymmetical relationship between independent variable (pre-image turtle) and dependent variable (image turtle) that characterises a functional approach to reflection is not made evident – indeed part of the attraction of the turn-swopping abstraction as far as students were concerned was the symmetry in the symbolic code produced during its MTG concretion. It was the task constraint forbidding communication with the red turtle that allowed students to experience in action how the position of an image turtle was dependent on that the pre-image turtle. This also helped students to shift their attention towards interfigural analyses and provoked connections of turtle behaviour to emotional as well as spatial connotations.

While all students developed general methods, not all students formalised them into procedures with variables. By this stage in the learning systems, students had been faced with a number of different views of variables. In the first task, the variable stick-person had introduced variables as generalised numbers – students could change the orientation of arms and legs by altering a numeric input to a turn. A second experience of variables came in the form of outputs of the `remember` tool, in which case they represented values dependent on the changing numeric inputs to the moves and turns that defined any example. The third experience of variables was associated with the communication tools, where turtles serve as inputs. Given this complexity, it is quite understandable that only some students turned the command sequences they developed into procedures incorporating the remembered variables. Others controlled some aspects of generality by manually changing individual commands, expressing it in action rather in any symbolic form.

The evolutions in knowledge of reflection as associated with this task are shown in Figure 6.32.

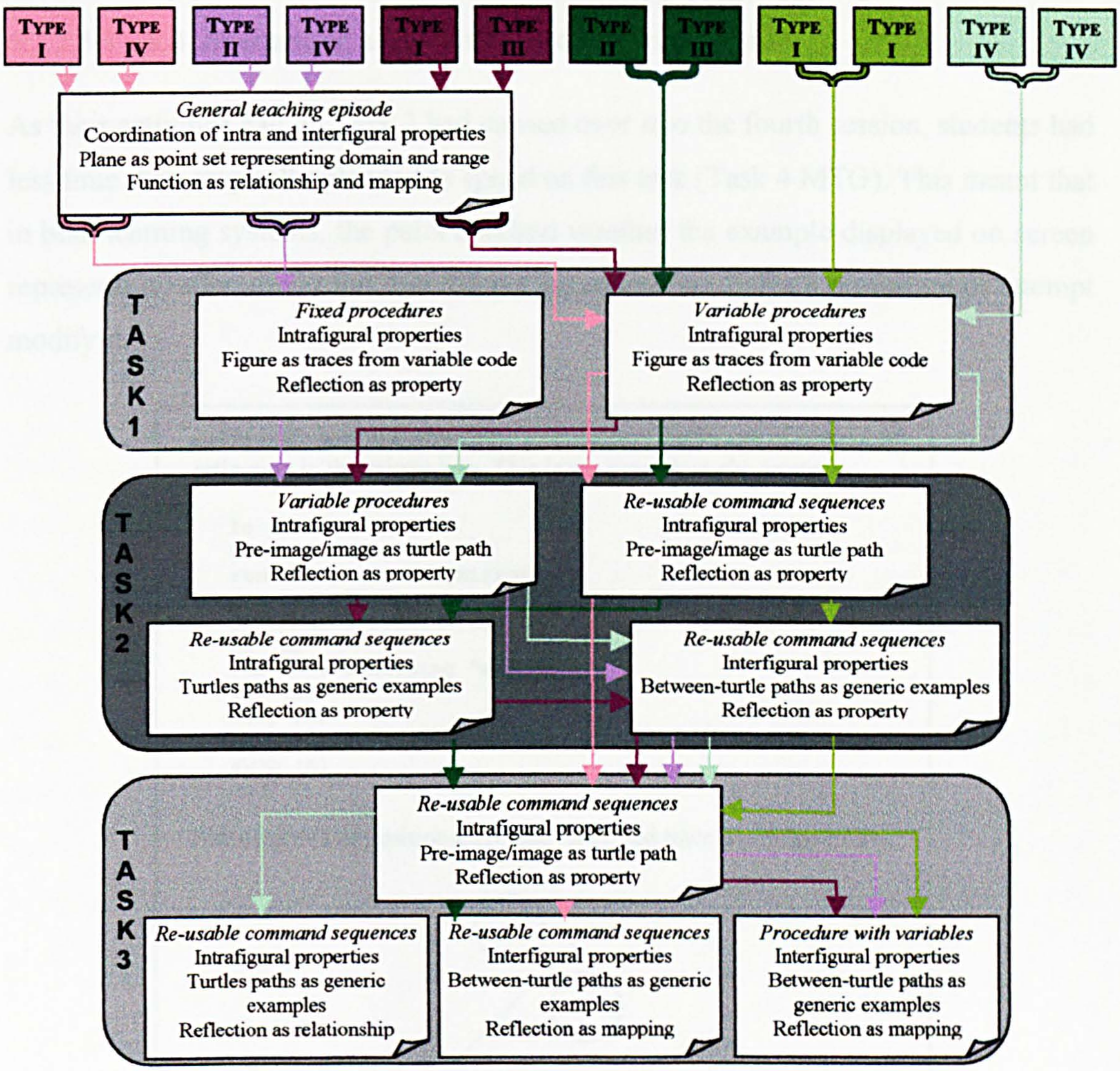


Figure 6.32: Student trajectories through the MTG learning systems to the end of third microworld task

Microworld evolutions

In terms of the evolution of the microworlds, three of the pairs had further extended the tool-set, writing a reflection procedure they believed to be general. The tools they had created were added to the remaining microworld files so they could be used in subsequent activities. The other students had not formalised their actions into procedures; nonetheless it was felt important to record their methods. Direct drive command sequences were copied onto the procedures page of the remaining microworld challenges files, making them available as a resource from which students could abstract commands, or sets of commands, if students so chose.

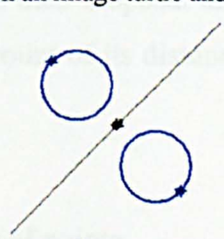
6.2.2.4 Task 4: Interpreting a formalisation of a transformation

As their activities had on Task 3 had carried over into the fourth session, students had less time than originally planned to spend on this task (Task 4 MTG). This meant that in both learning systems, the pairs checked whether the example displayed on screen represents a reflection or not, but did not go on analyse Zelda's procedure or attempt modify it.

Zelda built her own procedure to construct the image of a point by reflection in the mirror line. This is the procedure she wrote:

```
to zeldas
blue, hatchhere
remember towards "mirror
run :m1 rt 180
remember distance "mirror
bk :m2
remember toheading "mirror
run :m3
swop :m3
bk :m2
swop :m1
end
```

She used this to position an image turtle and trace an image circle.



Check whether Zelda's procedure produces the correct image-turtle always, sometimes or never.

If necessary, write a modified version of the procedure for Zelda.

Task 4 MTG: Zelda's procedure

Main strategies

Because of the fact that turtle states involve headings as well as positions, on the opening screen, the pre-image and image turtles both had headings parallel to the mirror turtle, but were facing in opposite directions relative to each other. All six pairs immediately concluded one was not the image of the other by reflection. The image turtle was argued to be "the wrong way round".

In addition to responding to this visual sign, four of the student pairs demonstrated the invalidity of the example by showing how both blue turtles could be turned towards the mirror turtle by executing a turn of `rt 90`, thus violating the swopped-turns property of symmetrical configurations.

Between-pair variation and variation associated with instructional approaches

There were no differences in the interactions of between pairs that could be interpreted in terms of the student types or the instructional approaches.

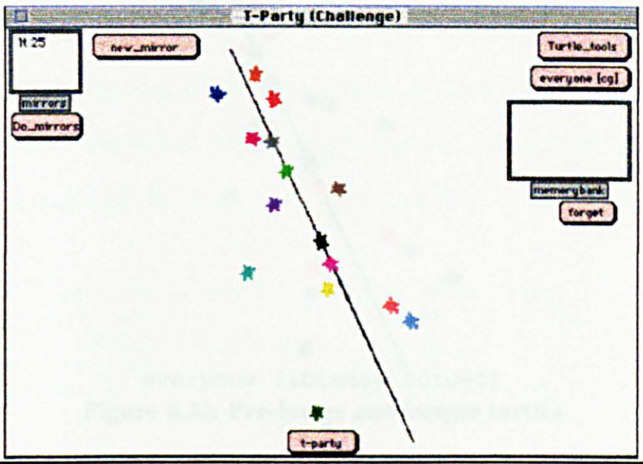
Constraints and affordances

Students interacted minimally with this task, neither expressing any knowledge that had not been used in previous tasks nor evolving any new approaches to reflection. They also built no computer constructions of their own. Their interactions do make clear that, whilst working with the microworld, all students could see that an image turtle in a correct location but with the wrong heading did not represent a reflective image. This stands in some contrast to the responses given, especially by the type IV students, in the first of the learning system activities, the paper-and-pencil test. It seems that with the MTG tools at their disposal all students had access to some way of judging an image that took account of its distance and orientation in relation to the other objects on screen.

6.2.1.5 Task 5: Reflection of a set of points

The final MTG task (Task 5 MTG) presented students with a screen of turtles to reflect. The task aimed to stress how a function for reflection should apply to all elements of the plane and not just to particular turtles or their paths.

Reflect the set of turtles on your screen in the mirror line.



Task 5 MTG: T-party task

Main strategies

Three of the pairs had access to the reflection procedures that they had written during the third task, while the microworlds with which the others were working contained the command sequences used to construct image turtles. All students made some use of their previously written code as two main strategies emerged. One involved using the student-defined reflection procedure and applying a reflection to all the visible turtles on screen using the `everyone` command. The other was to re-use command sequences, working through them one by one for each turtle in turn, making any changes manually.

Of the three pairs who had a reflection procedure, only the one developed using the mirror-turtle method for reflection actually produced the correct reflective image for all turtle states. Figure 6.33 shows the screen design output when this procedures was executed using the `everyone` command.

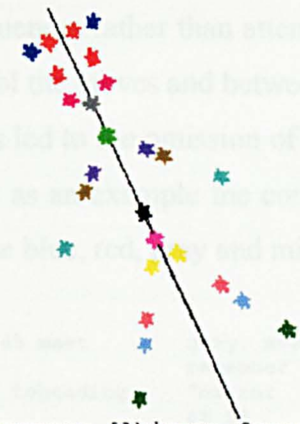


Figure 6.33: Pre-image and images turtles

There were several problems with the generality of the procedures based on the meeting method for reflection, which led students to use them as command sequences instead. First, the procedures did not work at all if the pre-image turtle had the same heading as the mirror turtle, since the two did not meet. This had been expected to produce a perturbation amongst the students, but none was experienced. Instead the problem was avoided by simply turning a 'parallel' turtle a little before attempting to construct its image.

A second problem was that depending on whether the image turtle moved forwards or backwards to meet with the axis of reflection, the last command of the sequence varied: when the turtle moved forwards, then it needed to move forwards and then rotate through 180° on the other side of the axis; when it moved backwards to meet the axis, the final turn was not necessary (this is illustrated in Figure 6.34).

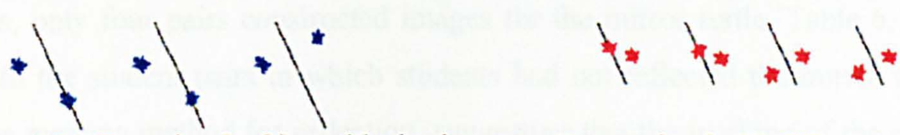


Figure 6.34: Turns involved in positioning image turtles associated with different ways of meeting the axis

This was not difficult for the students to see, but would have been very difficult to formalise into a procedure. A third problem was that, because of the way the meet tool had been defined, the meeting method could not produce an image for the mirror turtle – nothing would happen if a mirror turtle was sent to meet itself. Those who wanted to reflect this turtle had to find another way to do so.

By re-using the command sequences rather than attempting to work with procedures, students could manually control the moves and between-turtle communication needed for each turtle. Sometimes this led to the omission of commands associated with zero values and Figure 6.35 shows as an example the command-sets used by one pair to position the image turtles of the blue, red, gray and mirror turtles respectively.

```
blue, meet "mirror
remember toheading
"mirror
rt 30
remember towards
"blue
rt 30
remember distance
"blue
fd 51
```

a: A turtle that moved backwards to meet the mirror

```
red, lt 45 meet
"mirror
remember toheading
"mirror
rt 45
remember towards
"red
lt 135
remember distance
"red
fd 21
rt 180
```

b: A turtle parallel to the mirror that moved forwards to meet it

```
gray, meet "mirror
remember toheading
"mirror
rt 56
remember towards
"gray
rt 56
```

c: A turtle that was on the mirror

```
mirror, hatchhere
```

d: The turtle that defined the mirror

Figure 6.35: Modifying command sequences for different turtles

The construction of image turtles step by step was an arduous task, although it helped that it was not necessary to type each command (along with the necessary commas and apostrophes) from scratch each time. Once the visual image began to emerge (or was seen on the screen of others), this provided a powerful motivation for students.

Between-pair variation

One factor that varied between pairs was the attention given to the mirror turtle. Although all the students had wanted to construct images for the turtles located along the axis, only four pairs constructed images for the mirror turtle. Table 6.11 shows that both the student pairs in which students had not reflected the mirror turtle had used the meeting method for reflection, suggesting that the working of the `meet` tool mediated the way they treated this turtle.

Strategy	Student pairs		
Meeting method			
No image for mirror turtle	Alissa and Helen,	type II-IV pairing,	MTG-FI
No image for mirror turtle	Candy and Laurel,	type I-I pairing,	MTG-FO
Mirror turtle image constructed	Lizzie and Aimee,	type I-IV pairing,	MTG-FI
Mirror-turtle method			
Mirror turtle image constructed	Hadley and Lorna,	type I-III pairing,	MTG-FI
Mirror turtle image constructed	Prija and Jodie,	type II-III pairing,	MTG-FO
Mirror turtle image constructed	Kerry and Sophy,	type IV-IV pairing,	MTG-FO

Table 6.11: Distribution of student pairs according to strategies used to construct image turtles and construction of the image of a turtle invariant under reflection

There was however, one pair who had constructed images using the meeting method and yet were adamant that this turtle should have an image, as Aimee's comment below illustrates:

> *Every turtle has its own reflection turtle with the same distance away from the mirror and the same angle, except for lefts and rights. This one has no distance away and no angle, but it still has its own reflection.*
>
> (Aimee, type IV student, MTG-FI)

Aimee's comment suggests she had abstracted from her MTG interactions a meaning for reflection that is closely related to the notion of function, in particular she spoke of how each element has its own image – even when both are the same – and about the interfigural relationship by which elements in the input set are related to their pairs in the output set. As such her situated abstraction differed quite markedly from the intrafigurally based descriptions typically offered in paper-and-pencil contexts.

Between-system variation

While there were no particular differences in the strategies used to reflect the turtle-set that could be interpreted by considering differences in instructional approach, during the interactions of one of the student pairs of the FI system, there was some indication that students were developing a view of geometrical objects as turtle-sets. This occurred as Hadley and Lorna, the type I-III pairing, whose use of the `everyone` tool had led them to complete the reflection rather quicker than the other pairs in the system, were given the task of making the stick-person more complete.

They decided to do this by adding more turtles to fill in the "holes" in various parts of its body, then reflecting each new turtle using their reflection procedure. Unfortunately, the microworld failed the pair as they reached the limit of the number of turtles and the message `out of space` when they attempted to reflect the fourth turtle added (Figure 6.36).

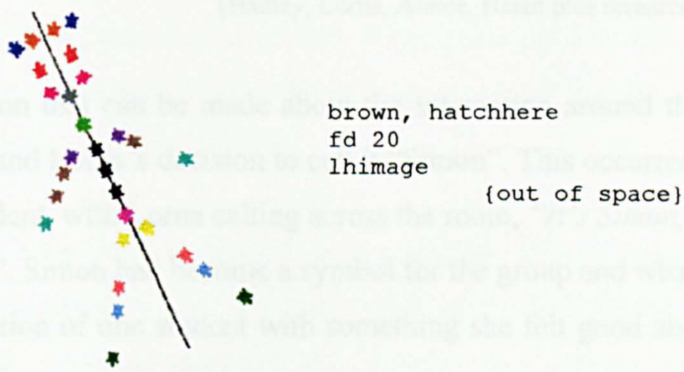


Figure 6.36: Trying to complete the person in the MTG-FI system

In the discussion that followed, they were joined by the researcher and a member from each of the other pairs to whom they communicated their view of their stick-person as turtle-set and of reflection as a relationship between sets. One of the other students also seemed to connect with the idea, although she argued that the figure be completed using turtle-traces, not because she could only think about figures in this way but because it would be more economical. Lorna's comment (in bold) in particular indicates a connection to the model of a two-dimensional world of the turtles that had been suggested by the researcher.

Res: It's run out of space. You can't add any more turtles. It's a pity. It was an excellent idea.

Hel: *What happened?*

Had: *We overloaded it, too many turtles, we wanted to fill all the holes with turtles and make their images, one there, one there, one there, one there and so on. But it's out of space.*

Res: It's not possible, there's not enough memory. It's a computer problem.

Lor: *Well, Simon will just have to be holey.*

Aim: *You could try joining him up.*

Lor: *Yeah, but for each one, find the distance, pen down, forward, back.* ***I thought we could make him all out of turtles. You know, because it is a world of turtles.***

Aim: *But still, each one you put, you need to put another one. Be easier to join it up.*

Had: *We could do that, there's not so many distances if you think about it.*

(Hadley, Lorna, Aimee, Helen plus researcher, MTG-FI)

Another observation that can be made about the interaction around the stick person related to Hadley and Lorna's decision to call it "Simon". This occurred as soon as its shape became evident, with Lorna calling across the room, *"It's Simon, have you seen Aimee, it's Simon"*. Simon had become a symbol for the group and what had begun as a personal association of one student with something she felt good about had turned into a kind of whole-group concept.

Constraints and affordances

Students' interactions during this task were mediated by the computer constructions built during Task 3 MTG. This meant that the relationships set up between turtles and their images were, generally speaking, the same as those used in the previous activity. However, in moving from operating on one turtle intended to represent the set of turtle-states to many turtles each illustrative of a particular turtle-state, this task raised some new issues to students and hence opened new windows onto their thinking about reflection as a function.

Although the visual configurations being produced on screen were very important to all the pairs, most of them appeared to be thinking beyond the production of a symmetrical design. This was particularly apparent in their desires to produce images for all turtles on the screen, including those along the axis. Students' activities as they constructed an image turtle with the same state as the pre-image were especially suggestive of a functional approach, with the idea of invariance under transformation connected to the enaction (or conscious decision not to enact) zero turns and distances.

The MTG tools and task also seemed to afford the construction of meanings in which geometrical objects are thought of as point sets, although this only happened in the FI system where this meaning had been emphasised by the researcher both in the teaching episode and in the introduction of the microworld tools. It may be that, unlike movement between intrafigural and interfigural analyses and the incorporation of aspects related to a functional approach to reflection, which emerged similarly as students interacted with the tools and the task of the MTG microworld, the connection of geometrical objects to turtle-sets will only be made in systems in which it is presented as a possible model to students.

Figure 6.37 below adds the knowledge expressed within the computer constructions developed during this task to the knowledge trajectories through the MTG learning systems, indicating how all students had incorporated aspects of a functional approach into the computational objects they worked with and some had extended the ways they thought about pre-images and images, so that they were seen as homogeneous sets of turtles, as well as traces on screen and turtle paths encoded in symbolic form.

Before leaving the discussion of this task, there is one other evolution to be described. The observations of students' interactions during this task not only suggested changes in students' thinking about reflection, but also afforded a change in the researcher's perspective. This was provoked by the analysis of the interactions of students who had incorporated the use of the `meet` tool in their image point constructions. Before this, from the point of view of the researcher, the goal had been that students would construct a general procedure encapsulating one construction method that could be applied to any element on the plane. Only one pair had achieved this in the MTG learning systems.

Analysis of the work of the other student pairs, however, indicated that they too were adopting functional approaches. It was just that their actions were suggestive of a function based on a series of if-then clauses – for example, if turtle is parallel, turn it then apply rest of sequence; if the turtle is the mirror turtle, hatch its image and do not apply sequence; etc. The students did not have the necessary fluency with the symbolic language of the MTG microworld to formalise the conditional relationships that had guided their activities, but did express generality in their actions. From this new perspective, the original vision of the student-generated reflection function seemed rather narrow – as long as the defining properties of the transformation are preserved, it is not necessary that the same set of procedures is used to map each element onto its image.

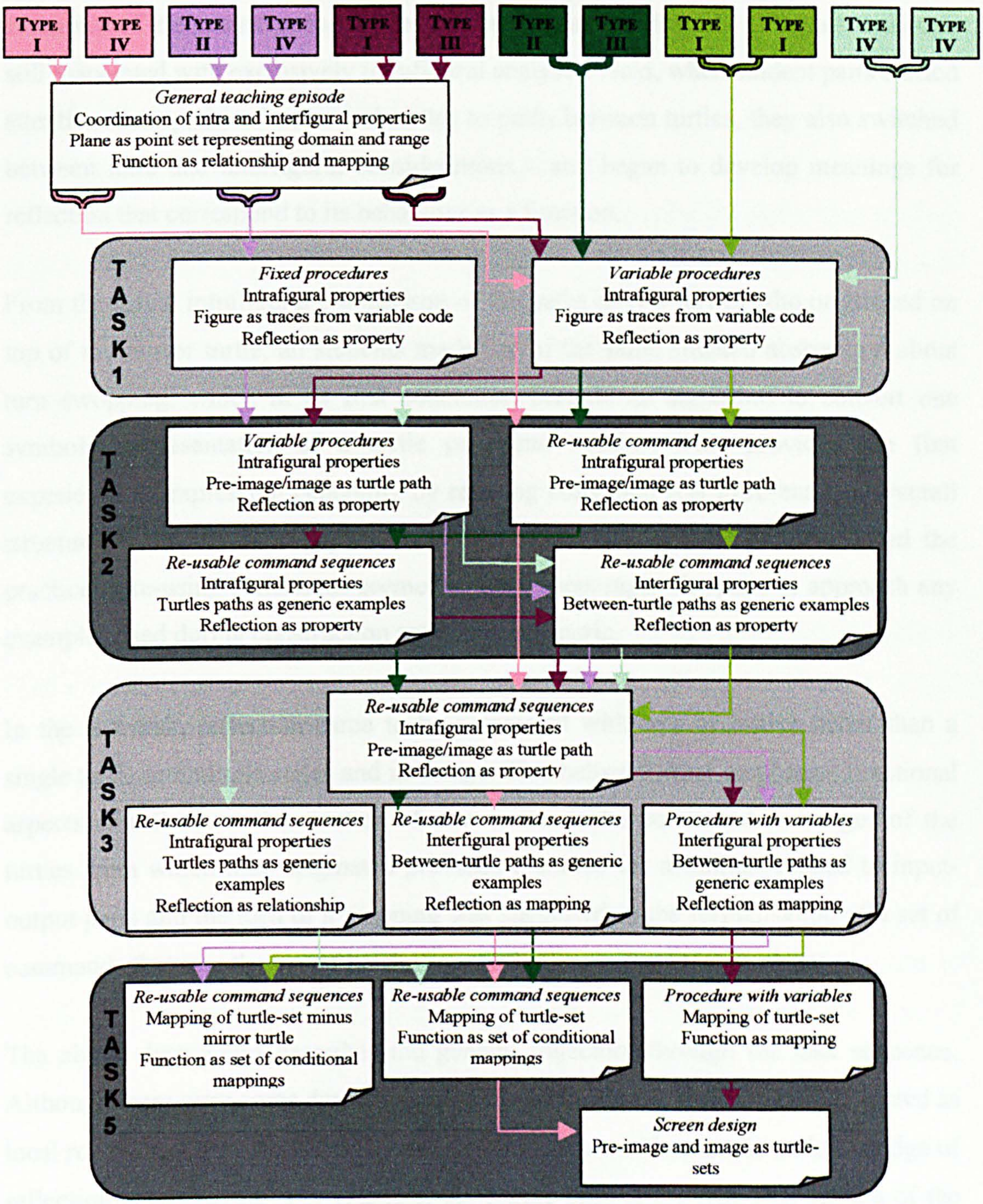


Figure 6.37: Student trajectories through the MTG learning systems to the end of fifth microworld task

6.2.1.6 Summary of evolutions in students' thinking about reflection during MTG interactions

There were some common features in the trajectories of all student pairs as they negotiated the demands of the MTG tasks. All of them developed ways of thinking about figures and then pre-images as the paths of individual turtles, in itself an extension of the meanings for figures evident in paper-and-pencil activities, although still associated with exclusively intrafigural analyses. Next, when student pairs shifted attention from paths of individual turtles to paths between turtles, they also switched between intra and interfigural considerations – and began to develop meanings for reflection that correspond to its behaviour as a function.

From the initial intrafigural comparison of the paths of two turtles who originated on top of the mirror turtle, all students made use of the same situated abstraction about turn swopping, which in its first concretion became an algorithm to convert one symbolic representation of a turtle path into another. This provided the first experience of expressing generality by re-using command sets to re-enact an overall structure, whilst modifying particular details of it. As the systems progressed the practice of re-using commands seemed to have encouraged the pairs to approach any examples used during construction activities as generic.

In the last task, reflection came to be associated with sets of turtles rather than a single turtle in multiple states and its image. This helped further emphasise functional aspects of reflection: the creation of new turtles to be positioned as images of the turtles from which they originated provided a source for meanings related to input-output pairs and the idea of a mapping was suggested in the formalisation of a set of commands that sent the newly hatched turtle from its initial to its final state.

The above description describes the general trajectory through the task sequence. Although there were some deviations to the main trajectory, they can be considered as local route changes rather than alternative trajectories – at least as far as knowledge of reflection is concerned. For example, the purple path of the type I-III pairing of the FI system in Figure 6.37 shows how this pair were more consistent than others in attempting to formalise methods into variable procedures. In this respect, their trajectory most closely matches that hypothesised in the design of the task sequence, and suggests the pair's goal was the same as that originally conceived by the researcher – the construction of a reflection function based on one general method applicable to all turtle states.

Another pair whose trajectory passed through the major ideas but followed a slightly different route was the type IV-IV pairing from the FO system (whose path appears in light-green in Figure 6.37). In their Task 2 interactions they seemed to be thinking about paths between turtles when they sent turtles to meet on the axis, but from the third task onwards, they developed a way of positioning image points that allowed them adopt a functional approach while concentrating on intrafigural properties of turtle paths.

Despite their emphasis on intrafigural properties, there is some evidence that their goal more closely matched that of producing a reflection function than the two pairs (shown in mauve and lime green in Figure 6.37) which followed the more general path until the final task when they chose not to reflect the mirror turtle: a choice that perhaps indicates an stronger concern with what the final symmetrical design looks like than ensuring all elements of the pre-image set have a corresponding element in the image set.

During most of the task sequence, there were very few between-system differences, although an important difference did emerge during the last task, in which students from the FI but not the FO systems extended their set of meanings for figures to include one in which they are seen as collections of turtles – a view that had been emphasised by the researcher in this system and brings students closer to the notion of mapping of plane onto itself, where plane also is represented as a homogeneous set of elements.

Apart from this one exception where student meaning could be traced back to a researcher-given abstraction, the other meanings related to reflection evolved from abstractions based upon the co-ordination of visual and symbolic resources available within the MTG microworld. Some students preferred to start from visual concerns and go on to connect these to symbolic representations, while other students began from the symbolic and moved to the visual, but because of the nature of the MTG tools it would have been difficult for students to avoid attending to both aspects.

Finally, the students in both MTG systems found ways of connecting with the microworld activities in ways that enabled them to bring a sense of themselves and aspects of their life outside the mathematics classroom to the learning systems. In both systems this helped students to engage in the task and to build their own stories in which they constructed a kind of anthropomorphic meaning for the activities of the turtles on screen. In this way, they seemed to want to build models of 'real-life' phenomena, not in the sense of expressing the mathematical properties inherent in an observed phenomenon, but by extending meanings of 'interpersonal' relationships by expressing them in terms of mathematical relations between turtles and their paths.

6.3 Beyond the microworlds

This section concerns the students' interactions during the paper-and-pencil based *final interview*, during which they worked individually with the researcher. The aim of these interviews was not to offer a 'before' and 'after' view of each student, which would be inconsistent with idea of using action-in-setting as the unit of analysis. Responses were hence not treated as objective measures of students' cognitive resources after working on the microworld tasks, rather the emphasis was on the meanings students tended to draw upon, given the changes in the mediation tools (from microworld to paper-and-pencil) and in interaction partners (from pair to researcher).

For the students in the two FI learning systems, the interviews followed the microworld tasks. In the FO systems, before the interviews, students participated in the general teaching episode (described in *§5.3.1*). During this episode, the two FO student groups constructed and discussed reflective images on paper-and-pencil, with the researcher guiding the discussion to emphasise the coordination of intra and interfigural properties and aspects related to a functional approach. No student descriptions of the planes or figures as point-sets had emerged during computer interactions in either FO learning system, therefore this aspect of knowledge was also not addressed during the general teaching episode – to do so would have imposed a general model in a way inconsistent with the FO approach.

6.3.1 Analysis of the final interview

Analysis is divided into two sections. The first considers students' strategies and responses when they were asked to apply a reflection to a plane represented with paper-and-pencil. The second discusses the extensions they made to their views on reflections and reflective symmetry.

6.3.1.1 Drawing reflective images

During this activity, students were asked to direct the researcher to draw the reflective image on paper and then check that the resulting figure was correct. This approach was adopted so that students had to make any mathematical properties explicit enough for another to execute them – in a way similar to making constructions explicit in computer interactions. The tools available were compasses, rulers and angle-indicators, of which only the latter two were used.

Seven students first directed the researcher to construct the image by constructing equal horizontal distances either side of the axis (Figure 6.38), producing an image correct only if a 3-D perspective is adopted. Five of seven rejected these constructions and went on to construct the correct 2-D image. Two students, however, were not able to identify any particular shortcomings of the reflective image nor attempted to produce an alternative, although they seemed far from convinced that their construction was correct. Both these students were those originally classified as type IV. Without external support – from microworld tools or another student – these two still had difficulties in distinguishing a reflective image from one produced by another isometry.

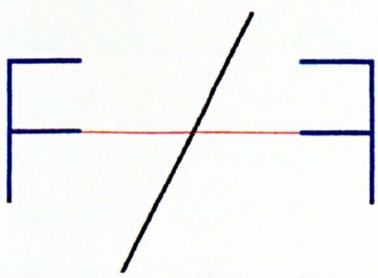


Figure 6.38: Treating the axis as vertical or adopting a 3-D perspective?

A number of different methods were used for the correct 2-D image, which varied according to the number of vertices on which students operated as well as the properties used. The methods are shown, along with the number of students who used them, in Figure 6.39. Three different ways of operating on the vertices in turn were observed (Figures 6.39a, b and c) along with three different methods for constructing the reflective image around one point (Figures 6.39d, e and f).

There were slightly more DEG students who operated on each of the vertices in turn than MTG students (eight as compared to five) with a reverse pattern among those who chose to build the reflective image around one point (three DEG as compared to six MTG students). There was an even greater difference in the strategies of DEG and MTG students as regards the properties they constructed. More of the DEG students used methods involving perpendicular constructions than MTG students (a total of ten of the DEG students constructed at least one perpendicular line, while only three MTG students included them). Again the reverse was the case for equal angle and distances properties, which were included in the construction methods of three DEG students and ten MTG students.

While responses varied according to which microworld students had worked with, there was a more even split between instructional approach both in terms of the way students operated on vertices (six FI students and seven FO students operated on them all, while five FI and four FO students constructed their image around one vertex) and the properties used (seven FI and five FO students constructed perpendiculars lines and six FI and eight FO students equal angles and distances).

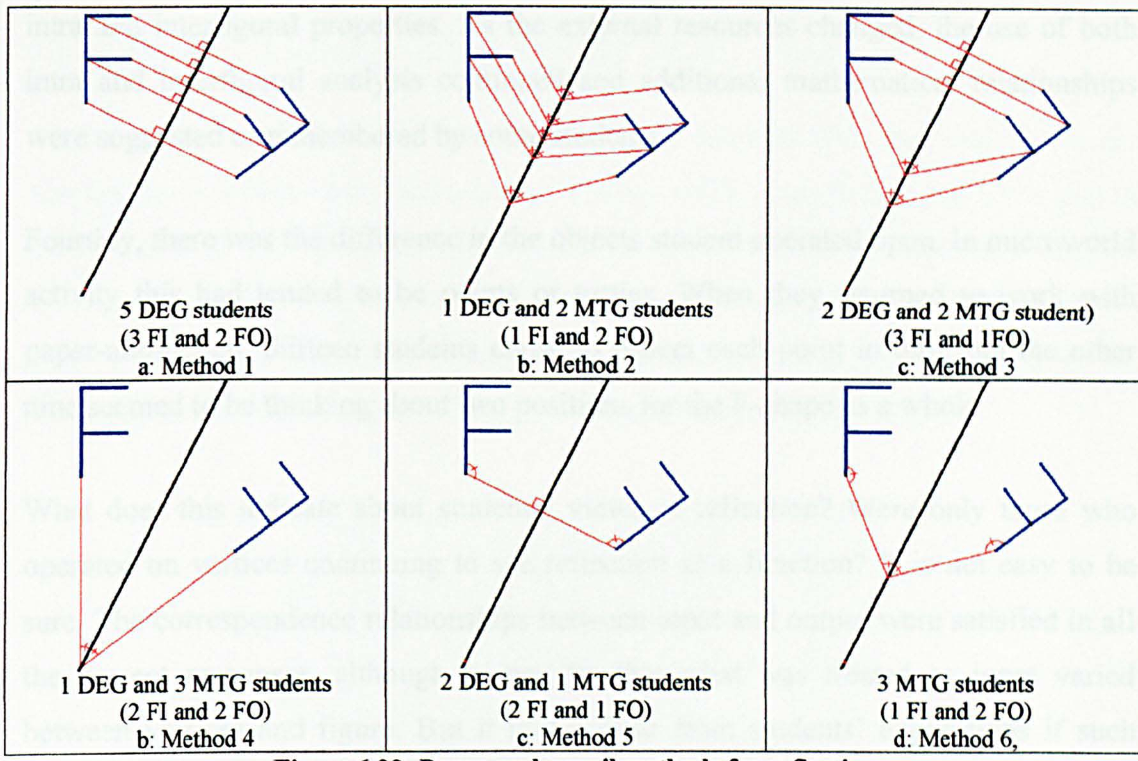


Figure 6.39: Paper-and-pencil methods for reflection

How might these responses be interpreted? The first point to be made is that during microworld activity incorrect reflective images were never accepted by students. In the absence of the microworld tools, on the other hand, some had difficulty in using the visual feedback on paper-and-pencil to evaluate their construction. Nonetheless, the majority were able to make use of interfigural properties to reject constructions that, although congruent, were not images under reflection. The second point, then, is that most students were analysing reflection from both intra and inter perspectives.

A third point relates to the properties students referred to: DEG interactions had seemed to emphasise perpendicular relationships and MTG interactions equal angle and distance relationships. In fact, the MTG students never explicitly discussed the perpendicular property during microworld activity but some of them did make use of it in this paper-and-pencil task. The tools of MTG seemed to have encouraged construction methods that did not call attention to this property, but this did not necessarily mean that students had no knowledge of it – only that they had not expressed this during MTG interaction. Students did tend to carry on using the same properties when they worked on the paper-and-pencil case, but perhaps more important than the particular properties they had learnt about was their attention to intra and interfigural properties. As the external resources changed, the use of both intra and interfigural analysis continued and additional mathematical relationships were suggested or remembered by some students.

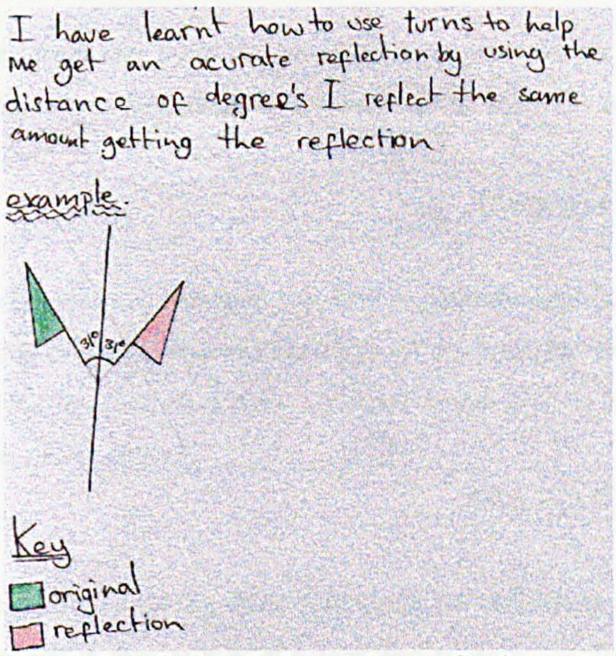
Fourthly, there was the difference in the objects student operated upon. In microworld activity this had tended to be points or turtles. When they returned to work with paper-and-pencil, thirteen students chose to reflect each point in turn, but the other nine seemed to be thinking about two positions for the F-shape as a whole.

What does this indicate about students' views of reflection? Were only those who operated on vertices continuing to see reflection as a function? It is not easy to be sure. The correspondence relationships between input and output were satisfied in all the correct responses, although it may be that what was treated as input varied between vertices and figure. But it is not clear from students' expressions if such functional relationships characterised the way they were thinking about the problem or not. The window onto their thinking-in-action was less clear during this particular activity than it had been during the microworld activities, partly because students had not been discussing their ideas with a partner but also because of the nature of the activity and especially its goal.

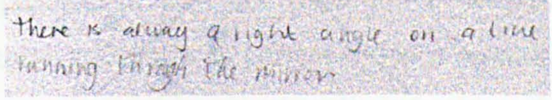
Constructing reflections in both microworlds had come to be associated with visual products on screen and the construction of tools to produce them. In the paper-and-pencil case, the only tangible product required was the visual product – the students did not build simultaneously any external resource that could be reused to enact other reflections. It did seem, though, that as they had been working on the construction of these external resources during microworld interaction, the majority of students had also extended the set of internal resources by which they made sense of reflection and could make use of some of the new ideas incorporated beyond of the setting in which they were originally constructed.

6.3.1.2 Students' descriptions of reflection and reflective symmetry

After they had completed the construction task, students were presented with the description of reflection and reflection symmetry written before they had taken part in the microworld activities. All students extended rather than changed their original descriptions. Many referred to angles (ten of the DEG and all of the MTG students) and Figure 6.40 presents two examples of written extensions, one referring to the equal angles either side of the axis and the other mentioning right angles.



a: Alissa's written description of what she learnt about reflection and reflective symmetry (type II student, MTG-FO)



b: Anju's written description of what she learnt about reflection and reflective symmetry (type II student, DEG-FO)

Figure 6.40: Written additions to students' descriptions of reflection and reflective symmetry

In both the figures above, the angles referred to interfigural relationships. Most students made some reference to interfigural properties, although eight continued to describe reflection entirely in terms of intrafigural properties, in which case they extended the set of properties common to pre-image and image, as illustrated in the excerpt below:

And I think I learnt that the two halves have the same angles as well, and the same area and, and they move the same.

(Kylie, type IV student, DEG-FO)

In other verbal additions to their original descriptions, six students made comments that can be contrasted with the views commonly provided by the inclusion of references to interfigural relationships (illustrated in the first comment below) or description of the operations by which a drawing can be transformed into its reflective image:

If you draw a line from any point on an object to a point on the image at ninety degrees through the mirror line, then the distance will be the same and the angle will be the same.

(Anita, type I student, DEG-FI)

When you make a symmetrical drawing, you have to measure anywhere along the line from on the drawing and then measure the angle on the line. And then measure it on the other side of the line and then draw the same length line on that angle. Then measure the angles on the shape.

(Prija, type II student, MTG-FO)

These comments suggested that the three meanings for reflection expressed in paper-and-pencil descriptions at the start of the learning system – reflection as perceived object, reflection as physical process and reflection as property of an object – had been joined by two more: reflection as correspondence relationship and reflection as mapping. Both these new meanings can be aligned with a functional approach to reflection.

Finally, the comment of one student suggests a reorganisation of the internal resources she used to think about reflection:

I made a bit of a mess of it before, because I didn't know there was a difference with symmetry and reflective symmetry, I thought it could just be the same.

(Maia, type IV student, DEG-FO)

This comment is suggestive of a transfigural perspective: reflective symmetry has become a distinguishable member of a larger group of symmetries. The activities of the learning system had not been designed to extend into consideration of the transfigural – this had been felt to be an over-ambitious aim – but, in order to understand the discrepancies between the images she constructed at the beginning and end of the learning system, this one student seemed to have been provoked to think about reflection's relationships to other transformations.

Figures 6.41 and 6.42 complete the schematic summary of student's trajectories through the DEG and MTG systems respectively.

Figure 6.41 shows that, among the five DEG students who gave final descriptions of reflection that were exclusively intrafigural, four had followed trajectories during which interfigural properties had rarely or never been formalised using a DEG construction tool other than `symmetrical point`. In contrast, both the students making references to functional aspects had worked in pairs where robust construction of reflection had been defined, as had the student who appeared to be thinking about transfigural aspects of reflection. Clearly, this does not mean that some of the students knew nothing of the interfigural properties, but perhaps those who constructed them only in action were not so aware of the properties as those who had formalised them into functions of their own.

In Figure 6.41, no obvious relationship between the type of computational object built (command sequence or procedure with variables) and the views of reflection expressed during the interview can be identified. The manual controlling of some aspects of generality during MTG interactions may have been associated with a greater synthesis between the visual and the symbolic than was the case for DEG interactions. There is, at least, no indication that re-using command sequences led to less awareness of particular properties of reflection than the use of procedures. One other observation is that the four students who referred to functional aspects during the interview were the type II and the type III students, the systems' perceptual and theoretical students. Again, this may be linked to a co-ordination of visual and

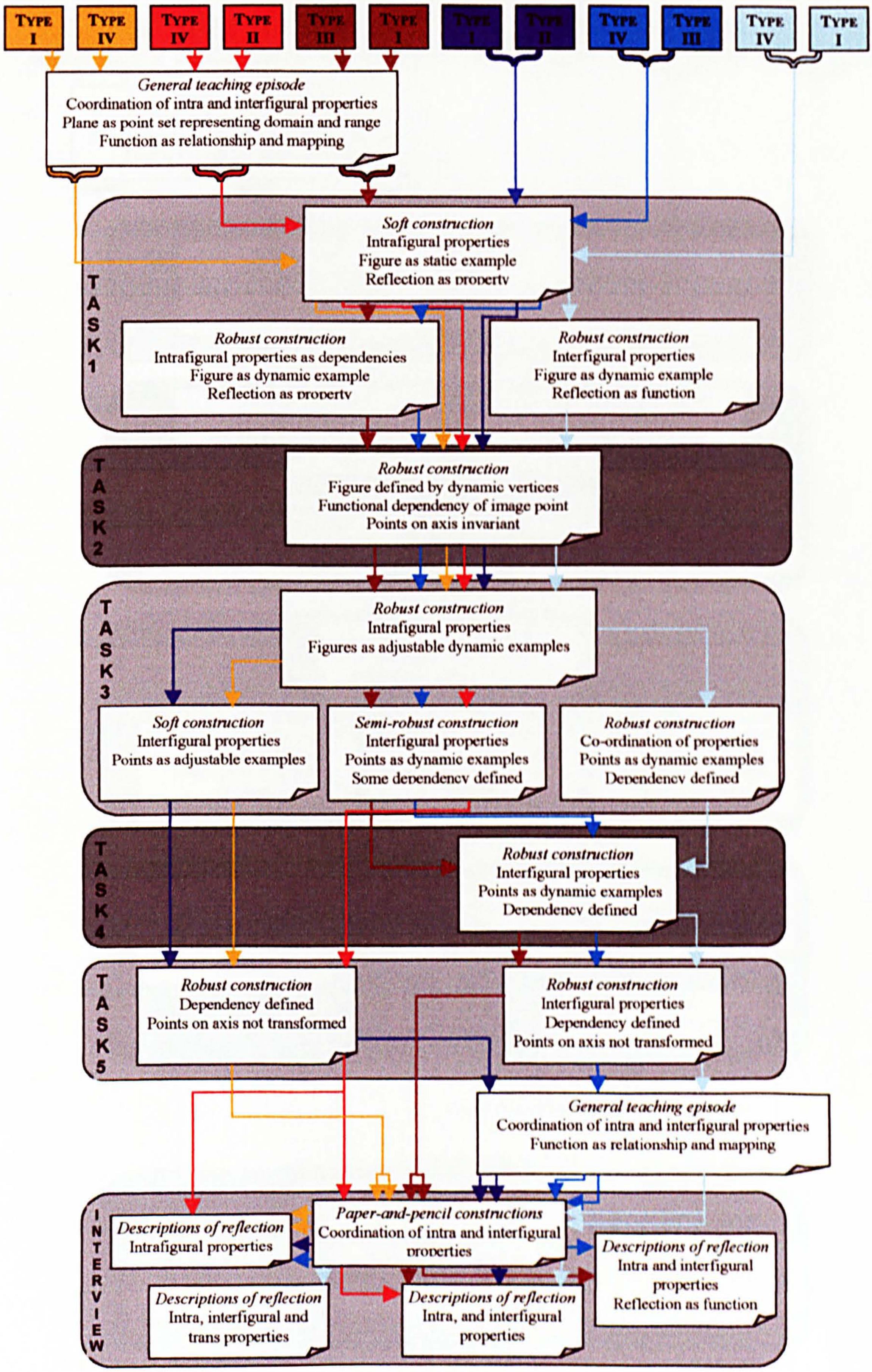


Figure 6.41: Student trajectories and knowledge expressed in DEG learning systems

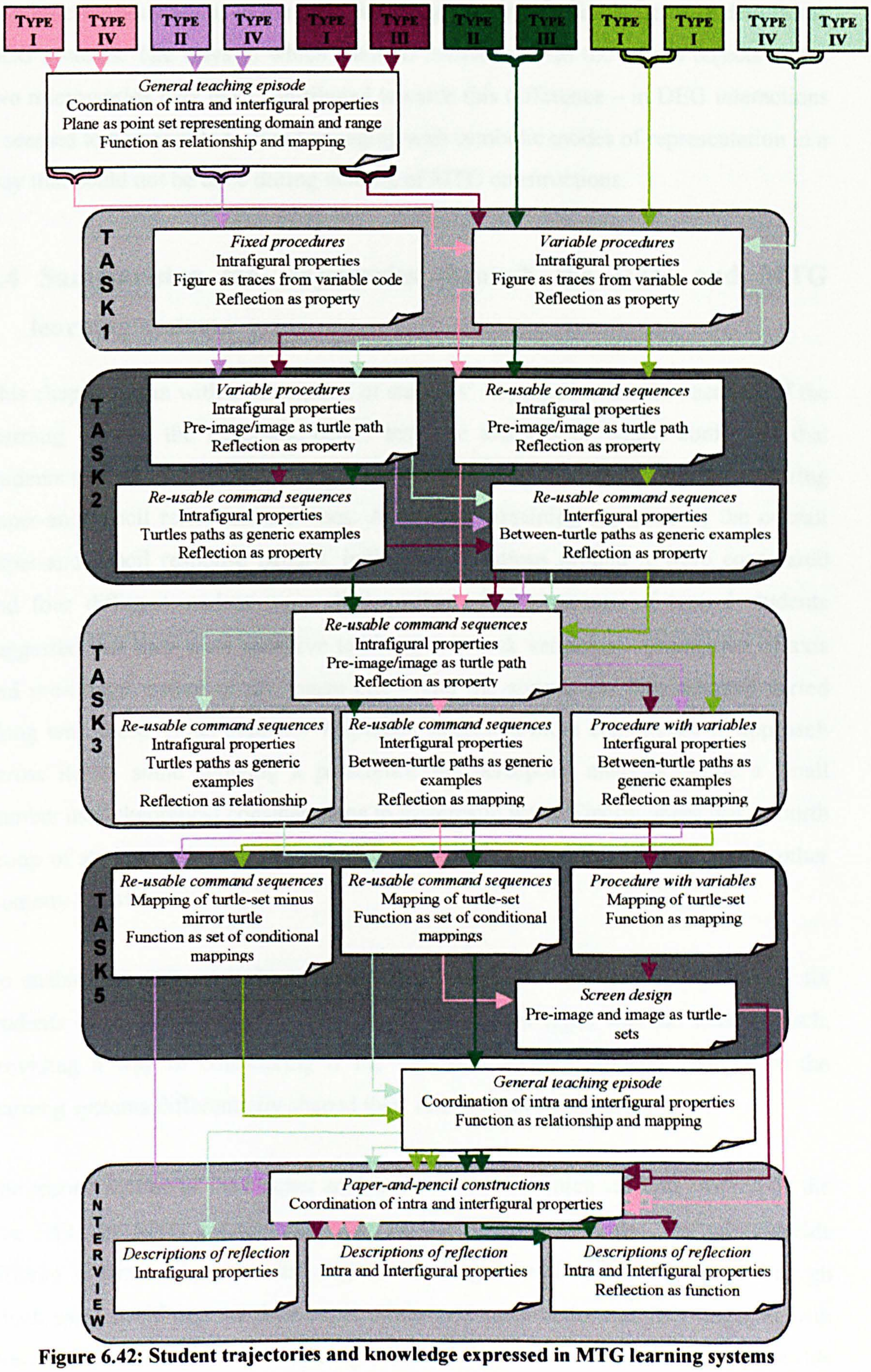


Figure 6.42: Student trajectories and knowledge expressed in MTG learning systems

symbolic representations in the MTG learning systems of a kind not so evident in the DEG systems. The ways in which students interacted with the screen objects in the two microworlds may have contributed towards this difference – in DEG interactions it seemed to be possible to avoid engaging with symbolic modes of representation in a way that could not be done during making of MTG constructions.

6.4 Summarising the trajectories through the DEG and MTG learning systems

This chapter began with a description of students' responses to the first activity of the learning system, the paper-and-pencil test, the analysis of which confirmed that students tended not to adopt interfigural perspectives or functional approaches during paper-and-pencil reflection activities. As well as obtaining a picture of the overall paper-and-pencil response pattern, individual variations around it were considered and four different student types distinguished. The responses of typical students suggested that they were sensitive to changes in task variables – orientation of axis and pre-image, nature of pre-image etc. – and the approaches they adopted varied along with these. Other students' responses suggested more consistency in approach across items: some showing a preference for perceptual analysis, while a small number used theoretical considerations in systematic ways. Finally, there was a fourth group of students who seemed particularly inclined to confuse reflection with other isometry transformations.

To enable comparisons to be made across the four learning systems, groups of six students were composed so that the profile of student types was the same in each, providing a way of considering if the internal resources students brought to the learning systems differentially shaped their interactions within them.

The second section of the chapter analysed the ways in which students worked on the five DEG or MTG microworld tasks of the learning systems. The microworlds differed in terms of their underlying models of geometry and in the means through which users could express their explorations and constructions as they engaged with these models. The aim of comparing students' interactions with these microworlds was to consider whether the processes by which knowledge develops within them were essentially similar or radically different.

A consideration of the interactions of student pairs as they worked through the microworld tasks suggests that the answer to this question depends upon the level at which the interactions are compared.

At the top level, interacting with either of the microworlds appeared to afford learners to experience and express knowledge associated with functional aspects of reflection and to encourage the incorporation of some interfigural analyses into their interactions. This occurred as students' goals in both DEG and MTG systems converged with that of the designers: that they construct a computational object that embodied a necessary and sufficient sub-set of the properties of reflection to produce valid visual configurations. One factor that had been similarly important in communicating this goal was how the tools and tasks of both microworlds had afforded views of particular cases as generic examples – in the DEG systems, dragging had played an important role in supporting this vision, whereas the MTG students had come to see both turtles and their paths as representative elements of more general sets and had been helped to do this by the use of symbolic command sets as templates. At this level, then, it could be argued that their knowledge had evolved in similar ways.

When the particular aspects of function and the specific interfigural analyses are compared, however, the results suggest that rather different knowledge-sets were evolved according to which microworld was used.

DEG interactions emphasised the notion of functional dependency, which could be defined through the use of construction tools and 'seen', under students' control, during dragging activities. The concurrent movement of pre-image and image point on the computer screen suggested view of function as correspondence relationship. The geometrical properties used to construct this relationship varied across pairs. Most common was to describe the relationship between pre-image and image point in terms of their equal distances from the axis along the perpendicular line through the axis by which they were joined. Other ways of describing the relationship were also identified, most notably by one pair who constructed a total of five equivalent construction methods.

As a result of their microworld interactions, students came to see the goals of the tasks as the production, on screen, of dynamic symmetrical configurations. Some students quickly appropriated this goal and the role of construction tools in defining the dependency relationships behind the dynamism of the screen design. Other students were inclined to use the dragging facilities to set up familiar properties by eye, with the result that particular instances of reflective symmetry could be obtained but the figure would mess up when dragged. There seemed to be two factors motivating the expression of generality in action, suggestive of different appreciations of the distinction between drawing and constructing.

First, some, often the theoretical type III students and those working with them, quickly appreciated the difference between drawing and construction but were not always sure which construction tool should be used to construct which property – and, especially, found it hard to construct equal distances. Providing exemplar symbolic descriptions of robust constructions helped these students. Second, the perceptually inclined type II students and their partners were among those who showed a tendency to think in terms of drawing only. For these students the provision of the `symmetrical point` tool was important – it was through this that they first experienced geometrical dependency. At the same time, though, this tool gave them a way of avoiding the formalisation of particular properties.

During MTG interactions, it was the notion of reflection as a mapping rather than as a correspondence relationship that tended to be emphasised. The dependency of the image turtle on its pre-image was stressed not by any concurrent movement on screen, but instead in the construction process, as most students thought of the process of reflecting as the construction of a path from the initial pre-image turtle state to the final image state. For one pair, however, the dependency relationship was expressed in terms of the intrafigural relationship between turtle paths – the path of the image turtle depended on that of the pre-image.

The properties that students used when building reflections, though expressed in different ways, were uniform across pairs and based on the use of the axis as an angle bisector – rays making equal angles either side of the axis and equal distances along them were constructed by all.

The goal for the MTG students became the construction of re-usable command sequences – either for tracing pre-images and images on screen or for defining the paths between them. As in the DEG learning systems, students found ways to resolve the problem of expressing generality without formalising it using MTG tools. Neither the DEG soft constructions nor MTG command sequences without variables had built in generality. There is a difference in that the practice of varying models in symbolic codes models the function of the variable it replaces, while, when a figure is dragged to satisfy a particular property, the process by which the property can be constructed is not necessarily modelled – as the representations of equal distance properties illustrates.

Another difference between the microworlds was in the reference systems through which students attempted to make sense of the interfigural. During DEG interactions, students constructed situated abstractions based on relative location and motion in a two-dimensional physical space. Relative location and motion were also central to the situated abstractions constructed during MTG interactions, but, unlike the DEG students, it appeared that the students' identifications with the turtle provided a further means of cementing together the relationships they abstracted.

In comparison to the mediation of the learning systems by the microworlds, instructional approaches had a less consistent impact on knowledge evolution. There were times in all four systems when students made use of theoretical aspects of their internal resources to interpret the empirical data on screen and other times in which the theoretical knowledge embedded in the microworlds produced regularities in the empirical data that the students could isolate. There were some students who showed more of a tendency to adopt theoretically driven approaches and others who appeared to favour data-driven styles of problem-solving. However, the task demands associated with both microworlds seemed to encourage a switching between theory and data, such that students used what they knew in ways that were not fixed but varied from situation to situation.

Through the situated abstractions that emerged from cyclical considerations of theory and data, students connected with the ideas about function and with interfigural relationships that were embedded into the microworld tools and task – for example, functional dependency became almost tangible in the DEG microworld and interfigural relationships between turtles in the MTG microworld could be played out on screen. Perhaps it was because of this that whether students were introduced to this knowledge before or after they embarked on microworld activity, or how they were introduced to the microworld tools, seemed to make little difference to the ways they approached the tasks. There were two exceptions to this general pattern.

First, in the DEG-FI system, student pairs invariably chose to enact equal perpendicular distances when attempting to construct reflections of their own, while there was a greater diversity in the relationships considered in the DEG-FO system. Second, students in the MTG-FI were the only students in any of the four systems to express views of geometrical objects as 'point'-sets (actually turtle-sets).

The first finding suggests that knowledge stressed by the teacher (researcher) can be reinforced through direct association with a microworld tool. The result may be a kind of funnelling effect, in which students are encouraged to adopt a particular way of thinking about reflection. This may help them to interpret particular relationship, but also runs the risk of implying that this relationship is the 'best' or even the only possibility.

The second finding presents a rather different case. While interfigural relationships tended to be constructed and functional approaches adopted whether or not they were specifically introduced into the systems by the researcher, the idea of geometrical objects as point-sets did not. This may be because interfigural aspects and functional aspects were embedded in the tools of the system in a way that point-sets were not. For instance, to use the `symmetrical point` tool, students had to specify the inputs to produce the intended output and they could then see how the latter depended on the former. In contrast, none of the tools of either microworld demanded that the students treat geometrical objects as point-sets. Perhaps it was for this reason that students connected to the idea only when it was introduced to into the system by the researcher.

But why did only those students working with the MTG microworld connect with this idea? One possibility relates to the epistemological status of the objects representing points in each microworld. In the MTG microworld, turtles were behind everything that appeared on screen and all turtles were controlled in the same way. Points in the DEG microworld were of three kinds: with basic points only one of a list of the basic objects that could be created on screen and constructed points treated as dependent upon, rather than constitutive of, other objects.

Finally, turning to students' last activities in the learning systems, when they returned to work with paper-and-pencil, results suggest they continued to engage in interfigural modes of analysis – generally, but not exclusively, re-using the same between-figure relationships – that had been enacted during microworld interactions. However, without the goal of constructing an external tool for reflections, it was the properties of reflection rather than its functional aspects that tended to be emphasised by most students.

Chapter 7

Conclusions

7.1 Background and Aims

The aims of the thesis were to design learning systems in which students' knowledge of the reflection transformation is brought closer to socially-accepted, institutionalised mathematical knowledge and to compare how students' activities shape and are shaped by different forms of mediation. The learning systems were designed to set into motion thinking related to the transformation, reflection.

To address the dual concern of understanding the processes by which mathematical knowledge is constructed while building learning systems that would support engagement in these processes, the study was divided into two sequential phases, the design phase and the comparison phase.

During the first phase, four learning systems were iteratively designed, through a series of successive steps during which tools, tasks and teaching interventions were developed as students' activities with them were observed and analysed. The empirical work of iterative design followed from a consideration of knowledge mediation in the mathematics education literature, starting with a comparison of constructivist and sociocultural theories of mathematics learning. Both theories posit that personal meanings constructed during experiences with and upon resources in the external world transform, and are transformed by, the individuals' internal resources. This indicated that learning systems should involve learners in constructive mathematical practices during which connections between internal and external resources are forged into 'formal' expressions of meaning. This was to be achieved by interaction in computational microworlds.

To examine the evolutions in knowledge in relation to the expressive means available in the learning system, two computational microworlds were designed: dynamic-Euclidean Geometry (DEG) and multiple-turtle geometry (MTG). The microworlds presented learners with different models of geometry along with different means for interacting with them: DEG interactions involved direct manipulation of a model of the theoretical field of Euclidean geometry; MTG interaction involved the programming of multiple turtles, whose movements around a two-dimensional surface were controlled by symbolic code.

A major difference was identified between instructional theories drawing from constructivist perspectives and those guided by sociocultural ideologies, which related to the primacy assigned to the individual or the cultural in the learning process. Constructivist approaches emphasise a filling-outwards (FO) flow in which personal understandings are moved gradually towards institutionalised knowledge. A reverse filling-inwards (FI) flow of instruction described in sociocultural accounts stresses moving from institutionalised knowledge to connect with learners' understandings. Teaching interventions in this study were therefore designed to allow investigation of these two different instructional approaches: the FO approach aimed to develop general mathematical models from learners' activities; and the FI approach intended to support learners in appropriating general mathematical models previously introduced.

Each learning system comprised a set of activities – one paper-and-pencil, five microworld, a teaching episode and an individual interview – involving interaction with similar mathematical content. Mediation was varied in the learning systems in two ways: expressive means and instructional approach as described above. The question that guided the research during this phase was:

- *What knowledge, and in what forms, should be embedded into the expressive means and the instructional approaches of learning systems in order that students are supported in connecting the knowledge they have with the knowledge they are supposed to learn?*

In the second phase, an in-depth analysis of the evolution of the four systems was conducted, as a group of six 12-13 year-old girls, guided by the researcher, interacted within each system. The questions steering research within this phase were:

- *To what extent do the different expressive means and instructional approaches incorporated in the learning systems constrain and/or afford actions and formalisations leading to evolutions in knowledge?*

  More specifically, what different meanings for, and analyses of, the transformation reflection, its properties and the objects upon which it operates evolve as the systems in which they are constructed evolve?

7.2 Tools for analysis

The metaphors inwards and outwards were also used to rethink the processes of concretion and abstraction in mathematics learning. Concretion is treated as the building of personal connections, whereby learners express their own sense of an experience within a specific mathematisable situation (Wilensky, 1991; Noss & Holyes, 1992). Abstraction is regarded as some conscious appreciation by learners of the generalised relationships implied in their expressions (Mason, 1989; Noss & Hoyles, 1996). In this study, interactions were considered in terms of these two complementary processes: concretion in terms of developing a sense of relation and particularising; abstraction in terms of drawing away and generalising. The assumption was that the construction of mathematical meanings must involve both.

As the study progressed this dialectic between concretion and abstraction became a focus for attention and the construct of *situated abstraction* (Noss & Hoyles, 1996) was used as a tool for analysing it as evidenced in students' activities. Situated abstractions are defined as general expressions of mathematical ideas made concrete in the domain in which learners are working. In the design phase, tools were designed to support the expression of situated abstractions, which were then taken as the basis for designing other tools. In the comparison phase, situated abstractions offered a means by which to describe students' meanings as they evolved.

A second tool for analysis was drawn from the work of Piaget and Garcia (1989), who suggested that major mathematical ideas pass through an ordered sequence of epistemological levels, which also characterise the historical development of mathematical knowledge. They proposed an iterative cycling through three levels –

intra, inter and trans, whereby attention moves from internal relationships defining objects, to relationships between them, then to structures into which internal and external relationships can be organised. The transfigural level was not addressed during this study as the focus was the reflection transformation in its own right rather than the structured set of isometry transformations. Analysis therefore concentrated on movements between the intrafigural and interfigural levels. These two levels were used during both design and comparison phases as a means of classifying task demands and microworld representations, and as a way of interpreting students' interactions with them.

7.3 Findings from the design phase

The learning systems incorporated paper-and-pencil and microworld activities. All the activities aimed to encourage students to express their knowledge of reflection while at the same time support learning. But, in the design phase, the primary aim of the paper-and-pencil activities was to help the researcher to build a picture of the knowledge students could be expected to express in paper-and-pencil-mediated settings (i.e., the usual classroom situation), whereas the primary aim of microworld activities was to encourage students to extend their knowledge of reflection and to open windows onto knowledge-in-change.

7.3.1 Paper-and-pencil-mediated situated abstractions

Before participating in this research, students had engaged in a number of activities involving reflection both inside and outside of the mathematics classroom. The first step in the design process was to catalogue aspects of the intended knowledge students expressed during paper-and-pencil activity – i.e. their situated abstractions – and those they did not.

Analysis of students' responses to the various paper-and-pencil activities suggested that they made sense of reflection using the following meaning fragments:

⇒ Reflection is a perceived object (image), a physical process (that produces image) or a property of a symmetrical whole.

⇒ Constructing reflections is about constructing symmetrical designs that have two parts either side of a line that look the same.

⇒ The 'space' associated with reflection can be broken down in two sides of the axis, with whole figures on one side reflected onto the other side.

⇒ Both two-dimensional and three-dimensional perspectives can be adopted during the construction and evaluation of reflective images.

As students made connections between these meaning fragments, their activity appeared to be guided by three (overlapping) situated abstractions:

- *Reflection involves images that have the same size and shape as pre-images and are drawn opposite to them.*

- *Reflection involves figures that can be can be cut in half by drawing a line down the middle or by folding.*

- *Reflection involves reproducing all that is on one side of a mirror on the other side of it.*

Analysing these situated abstractions in terms of the intra/inter distinction suggests that students are engaging almost entirely with intrafigural concerns – identifying regularities through comparisons of internal properties of objects (pre-image, image and the symmetrical whole they compose). Both the abstractions and the emphasis on the intrafigural make sense when the tools commonly used to construct reflections are considered. Activities involving mirrors obviate any need to focus on between-figure properties – the mirror takes care of the distance of figures from the axis (as well as possibly bringing three-dimensional consideration into play). Similarly, folding provides a way of positioning figures at equal distances to a line without actually measuring any distances. Presenting figures on squared-paper can make their internal properties more easily reproducible. However, the use of squared-paper also highlights between-figure properties, such as parallelism, for example, that are not general to reflection and can encourage students to include extraneous relationships as they construct visual images of reflection. This last case, in particular, illustrates how the mediational effects of tools are not always positive for mathematics learners – and nor is the process of abstracting generality from sets of specific cases. The use of squared-paper appeared to constrain the students to work with a rather particular range of geometrical figures and axes of reflection. If they did not know what the reflective image should look like, it was also difficult to know which of the various properties highlighted in a visual representation should be attended to. Repeated experience with like examples is likely to confound any tendency to make generalisations based on properties embedded in the tool rather than the transformation.

One other effect of the tools associated with paper-and-pencil investigations of reflection was that they seemed to emphasise mathematical products over mathematical process, with the consequence that functional aspects of reflection were not expressed during students' paper-and-pencil activities.

The situated abstractions presented above are illustrative of the general tendency of the whole group of 12 and 13 year-old students who participated in this study. They are also consistent with findings from previous research projects where the role of the tools on knowledge mediation was not acknowledged (for example, Küchemann, 1981; Bell, 1993). The consistency of these results reaffirms the importance, when evaluating students' solutions, of taking into account the tools employed in their expression. This provides support for the idea that expressive means alter the flow and structure of cognitive activity that forms the mainstay of Vygotsky's theory of knowledge development (Vygotsky, 1981; Cole & Wertsch, 1996).

The general picture of responses to reflection activities indicated the tendency of all students to concentrate on intrafigural analysis. However, there were variations in the ways students constructed and evaluated symmetrical designs on paper. Some students adopted perceptually motivated approaches. They tended to draw the 2-D images correctly, but had difficulty in describing how they knew where they should be located; some students could describe the properties on which symmetrical constructions were based, but could not see that their intrafigural treatment might result in visually incorrect images; and some students were so fixed on the congruence of pre-image and image that they would accept almost any design as a reflection provided it had two equal parts. The typical students fluctuated between approaches, according to task features, sometimes evaluating visually, sometimes focussing on properties and sometimes just checking for equal parts. Taken together these results suggest a fragmentation of knowledge, with a lack of coordination between visual representations and the text by which they can be described. Rather than encouraging the synthesis between these two aspects into the figural-concepts Fischbein associated with geometry learning (Fischbein, 1993), the tools commonly used in schools mathematics may encourage their separation – with folding and mirrors allowing students to produce visual images without attending to geometrical properties and squared-paper emphasising properties without helping students visualise reflections.

In short, the tools through which school students usually encounter reflection, while helping them think about intrafigural properties, appear neither to encourage any synthesis between visual and symbolic aspects, support students in adopting interfigural analyses nor constrain them to think of reflection in two dimensions.

7.3.2 Microworld design issues

Findings associated with the design of the microworlds concerned general issues related to how their tools and tasks both shaped and were shaped by other aspects of the learning systems, and more specific issues involving learning the mathematics of reflection.

7.3.2.1 Tools and tasks to support students in mathematical meaning construction

Observation of students' work with the developing microworld tools confirmed the role of situated abstractions in mathematics learning: students' meanings emerged in interaction with the microworlds as students juxtaposed formal expression using microworld tools with experienced activity and visual feedback. The tools played an important part in enabling this juxtaposition – during students' actions with the microworld tools, visual representations and their underlying texts could be simultaneously constructed, narrowing the gap between action and formalisation that exists in paper-and-pencil situations.

It was also clear that students became engaged with activities into which they could 'import' experiences related to their identities outside the mathematics classroom (for example, dancing people and meetings of friends) to help make sense of the microworld tools and the mathematics relationships they could express with them. The tools of the microworlds were intended to act as *evocative computational objects* (Hoyles, 1993), objects that would 'evoke' the intended knowledge. Results from the design phase suggested that tools are most likely to be experienced as evocative when they are introduced in tasks that invite the creation of stories in which the computational objects built on screen represent a model of an imaginary situation, as well as a mathematical one.

It was found that students were highly sensitive to what might appear to be irrelevant details of tools, which meant that tool design involved rather more than simply deciding what mathematical properties students would need to represent and creating tools that could be used for this purpose. A tool's name, its place in the organisation of the system, the context in which it was used, as well as the feedback it provided were all found to be important. For example, changing the name of the `circle` tool and moving its location from the `creation` to the `construction` menu made it more likely to be associated with the construction of equal lengths (although this function was still not completely apparent). Similarly, by changing the name of the `make` tool in the Logo programming language to `remember` and the feedback it provided (a symbolic record of the variable and value created), its function became more transparent to students.

The process of naming tools was thus important in shaping students' interactions with them in both microworlds. From the designer's view, the naming process itself was shaped by aspects of the microworlds and the geometries they modelled. The DEG microworld was designed to model Euclidean geometry and the names given to the tools reflected this. For example, it was to stress its role as a construction tool that `circle` was renamed `compass` – the Euclidean geometry construction tools associated with transferring equal distances. The desire to constrain students to make use of the terminology associated with Euclidean geometry played a part in the design decisions about the naming of DEG tools.

In contrast, when designing the MTG kernel, there was a tendency to choose names that matched students' descriptions of the actions they performed rather than using conventional geometry terms. The names hence emerged in interactions between the designer, the students and the microworld based on Turtle geometry. For instance, the `meet` tool was built when it was observed how students wanted to place a turtle at the point at which the paths of two turtles met and its name was chosen to capture how they communicated about the activity the tool modelled.

Tools also gained meaning as they were used in tasks. The extent to which meanings corresponded to the knowledge the tools had been intended to evoke depended at least partly on the level of control students felt over the task-solution process. If the solution process was very highly directed and students told which tool to use, they felt little or no need to attend to the tool's role in shaping interaction. Similarly, breaking down a task into a series of specified steps, ensured that students used particular tools, but did not guarantee any real engagement with them (an example of the didactical paradox described in Chapter 2). Another problem was that if tasks involved the use of tools that 'took care' of mathematical properties, students sometimes enjoyed observing the visual effects, but were not necessarily provoked to construct any mathematical meaning for them (an example of the other side to the didactical paradox, called the play paradox by Noss and Hoyles (1996) that was introduced in Chapter 3).

These findings pointed to the importance of designing tasks in which the available tools could be used in diverse ways to illuminate geometrical structures and relationships without completely resolving all the task's demands. Tasks most successful in achieving this delicate balance were those that involved the students in the construction and validation of their own computational objects, objects that simultaneously produced the required visual products and contained a trace of the construction process.

7.3.2.3 Microworld tools for reflection

Turning more specifically to the construction of mathematical meanings for reflection, the analysis of students' paper-and-pencil mediated meanings indicated a need for tools that would enable them to extend their mathematical analyses from the intrafigural to the interfigural and attend to functional aspects, as well as the properties, of reflection. Following the task design criterion outlined above, these tools were to be introduced in the context of microworld tasks that engaged students not only in the construction of visual images of reflection but also the building of a new external resource, a computational object, to describe the mathematical properties underpinning their visual productions.

Microworld activities were then designed based on a hypothesised learning trajectory for reflection whereby students would first learn to re-express, both visually and symbolically, their intrafigural analyses using microworld tools then go on to revision their constructions from an interfigural perspective. In practice, the process of devising tools to support the intended evolutions in knowledge was complicated by epistemological differences between dynamic and turtle geometry.

With respect to the DEG microworld, the tools to express intrafigural properties were not easily appropriated by students. Perhaps this was because all DEG construction can be viewed as models of interfigural as well as intrafigural relationships: using them entails specifying the input objects and how they must be related to produce the output object. This meant that students had to think about the familiar intrafigural properties they associated with reflection during paper-and-pencil activities – especially the congruent properties they were used to ascertaining through metric rather than geometric means – in different ways. In contrast, the `symmetrical point` tool used to represent the interfigural relationship between pre-image point, axis and image point was easily appropriated into student activities, although it was clear that this tool was not always used with the particular geometrical properties it produced in mind. The purpose of the tool was clear, but its underlying structure was not. In order that students also engaged in the properties defining this tool, tasks that focussed attention on its process rather than its products were required.

For the MTG microworld, it was found less difficult to devise activities provoking students to attend to both intra and interfigural aspects of reflection. This was interpreted, at the end of the design phase, as a result of how turtle paths represented both figures and the relationships between them, thus blurring the distinction between intra and interfigural analyses. The major design challenge was therefore not designing activities but constructing tools by which students could reconstruct turtles paths and encode some generality into symbolic expressions representing them.

The different emphasis in design on tools and tasks in the two microworlds indicated another way in which choice of software shaped the design process and the relative ease to which students appropriated particular microworld tools questioned whether the relationship between intra and interfigural is necessarily sequential in the way Piaget and Garcia suggested. Although such a sequence might characterise how mathematical knowledge develops when mediated by paper-and-pencil tools, different ways of interacting with geometrical objects were made viable by the tools of both the DEG and MTG microworld kernels.

The final focus of the design phase was on instructional approach. This involved deciding upon how to present general paper-and-pencil models of reflection before task introduction (FI) or encouraging students to re-express their computational models in the paper-and-pencil context (FO). Two findings were associated with this final focus. First was the importance of ensuring that control for constructing solutions remained with the students during the microworld activity, so that they had the chance to make their own connections between their personal knowledge and the knowledge embedded into the systems. Second, whether models of general mathematical ideas were to be introduced to, or elicited from, students, this should be done in ways that stressed their connection to, not detachment from students' ways of thinking about reflection.

7.4 Findings from the comparison phase

The participants of the four learning systems were chosen on the basis of their responses to the paper-and-pencil test (the first activity of the learning systems). Each group of six consisted of students with similar response patterns: two were typical students, one was a student with a good perceptual feel for reflection, one was a student whose responses were theoretically based and two seemed to accept any geometrical configurations with two equal parts as reflections. In this section, the findings associated with the comparisons of the four systems are presented.

Overall, the comparisons suggested that the microworld tools and tasks had a more consistent impact than instructional approach on the ways meanings for reflection evolved in the learning systems, with less of an overlap in the learning trajectories of students following the same instructional approach than those using the same microworld.

7.4.1 DEG-mediated situated abstractions and the trajectories by which they evolved

Analysis of students' interactions with the DEG activities suggested they were using the following meaning fragments to make sense of reflection:

⇒ Reflection is a correspondence relationship between pre-image and image points.

⇒ Constructing reflections is about constructing dynamic figures that remain symmetrical under dragging.

⇒ The properties associated with reflection include: pre-image and image figures with congruent corresponding segments (and sometimes angles); corresponding pre-image and image points with equal distances (and sometimes angles) from any point on the axis; lines joining corresponding pre-image and image points that are perpendicular to the axis of reflection; and corresponding pre-image and image points that coincide on the axis of reflection.

⇒ Figures are dynamic two-dimensional objects composed of vertices and segments.

⇒ To reflect a pre-image, all points on the screen excluding those on the axis should be operated upon.

The meanings that evolved were not uniform across all pairs in that different students worked with different sets of properties in their attempts to position reflective image points. However, regardless of the specific properties used, the following situated abstraction characterises the way the meanings came to be connected in students' DEG expressions:

- *Reflection involves pre-images and their images. Pre-image points are moveable and each one controls its own image point, so that, as the pre-image point is moved, its image point moves equally closer to or further from the reflection line and the pair meet on this line.*

This abstraction is generalisable to all reflections, makes references clearly related to the resources of the DEG systems and can be connected to functional approaches. It was not however formalised by all students into a reflection function, as three different trajectories through the DEG tasks emerged.

Three of the six DEG pairs progressed through a largely perceptually based trajectory. Their engagement with any functional dependency was mediated only by the use of the `symmetrical point` tool. Other construction tools were avoided as students built any distance or angle properties by dragging – that is, they tended to build soft-constructions in which generality was expressed in action and not formalised. Both students who had exhibited a tendency to rely on visual concerns during paper-and-pencil activities followed this trajectory, along with two of the students who gave typical paper-and-pencil responses and two who had associated reflection with congruency alone.

Two other pairs progressed through a second trajectory, in which theoretical concerns were dominant in guiding activity and a reflection function was gradually formalised. In this trajectory, some access to a symbolic representation of constructions of angle and distance properties seemed important – and in the absence of such an external resource, students tended to resort (reluctantly) to the soft-construction of properties.

Both pairs involved in this trajectory included the students whose paper-and-pencil responses had suggested a prioritising of theoretical over perceptual concerns. In contrast to the paper-and-pencil case, their DEG interactions suggested a greater coordination of the two aspects, as the DEG tools provided visual feedback enabling the visual validation of theoretically motivated constructions. It was in the course of this trajectory that students expressed reflection in ways that closely resembled its traditional school-book definition (see, *§3.3.3*), building computer constructions based on the equal perpendicular distances of pre-image and image points to the axis.

A third trajectory was followed by one pair, who during the first DEG activity identified and constructed an interfigural property. Their interactions were characterised by flexible movements between intra and interfigural analyses and by the building of a set of equivalent robust constructions. Their work illustrated creativity in expression rarely attributed to students' interpretations of transformation geometry – even in studies involving dynamic geometry explorations of reflection, far greater attention is given to the perpendicular property of reflection than any other means of construction (see, for example, Guillerault. 1991; Laborde, 1995; Hölzl, 1996; Noss, Hoyles, Healy & Hölzl, 1994). This pair found a number of valid construction methods and, as they did so, formalised multiple connections between the ideas of reflection as function and property and synthesised theoretical and perceptual aspects of reflection. This creativity was made possible by the work during the design phase, since providing tools which connected with students' thinking, allowed students to construct reflection in ways that were meaningful to them.

This finding highlights the importance of taking account of students' ideas in the design of learning systems. Extended experience of a mathematical object from a particular perspective can lead to it being seen in rather specific ways – especially if it is always encountered in association with the same tool-set. In the case of reflection, this can lead to a tendency to constrain students to think of a particular reflection construction – generally the equal perpendicular distances construction that was expressed by the majority of DEG students. The activities of this one pair show that, given access to appropriate tools, there are a number of alternative (and equally valid) constructions by which reflection might be represented and which might be more meaningful to students than the conventional construction.

The work during the design phase had suggested that the structure of the DEG tools might have the effect of changing the relationship between the intra and interfigural. Their interactions provide some support for this conjecture. The computer constructions built by this pair resulted from a to-ing and fro-ing between analyses rather than the progression from a lower to a higher level of thinking that is suggested in Piaget and Garcia's descriptions.

7.4.2 MTG-mediated situated abstractions and the trajectories by which they evolved

Students' interactions during the MTG activities suggested they were evolving the following meaning fragments for reflection:

⇒ Reflection is a mapping from one turtle state to another or a relationship between the paths of two turtles.

⇒ Constructing reflections is about sending turtles to symmetrical locations and tracing symmetrical paths.

⇒ The properties associated with reflection include: in the paths of both pre-image and image turtles, the distance to and turns towards the line drawn by the mirror turtle are equal, although there is a reversal in the orientation of turns; and image turtles are on top of pre-image turtles along the line drawn by the mirror turtle.

⇒ Figures are two-dimensional turtle paths.

⇒ To reflect a pre-image, it is necessary to operate on all turtles on the screen.

The following two situated abstractions characterise how these meaning fragments came to be connected through (rather different ways) of focussing on turtle paths.

❑ *Reflection involves symmetrical turtle paths containing the same list of commands except that left and right turns are swopped.*

- *Reflection involves turtles with partners that they meet on the mirror line when they run forward at the same speed.*

The two abstractions above capture the sense-making activities of all students during MTG interactions. They were expressed in computer constructions as students moved between intrafigural analysis based on the identification of regularities in individual turtle paths to an interfigural analysis of the paths by which one turtle can be mapped onto another. This interpretation of intra and interfigural analyses in the MTG microworld represents a shift from that posited at the end of the design phase, where it had been framed by an essentially Euclidean view of figures instead of a turtle-geometry view.

Although all MTG students adopted interfigural analyses at some stage during microworld interaction, it became apparent that it was possible for students to construct a MTG reflection function while still adopting largely intrafigural analyses – one pair, for example, decided to write a function based on (re)constructing paths for both the pre-image and image turtle, rather than defining the path of one to the other. This finding suggests that the alignment between interfigural and functional approaches suggested by Piaget and Garcia may not be generalised to activities involving computational tools.

The MTG reflection functions expressed by all the students, regardless of their paper-and-pencil-defined type, shared one feature in common: the visual representations on screen were necessarily accompanied by their symbolic description. Some of the methods appeared initially motivated by visual considerations, with students indicating imagined paths on screen before thinking about the code to produce them. Other methods began with a focus on symbolic aspects – operations on the set of commands that had been used to position the pre-image turtle, for example. Whichever the starting point, once students had accepted the aim of writing a reusable general command-set, their computer products comprised a synthesis of the visual with symbolic: that is, visualisations were coupled with the symbolic code that instantiated them and both representations became, mathematical speaking, one and the same.

There were other common features in the trajectories of all MTG pairs. As the practice of re-using command sequences but changing the values within them evolved, turtle paths were treated as generic descriptions, with an overall structure that was maintained while particular features were altered to produce a range of specific cases. Although students made some use of variables in the computer constructions they built, this varied across pairs and across tasks, and the majority continued to express some of the generality in their methods through action not formalisation.

Finally, when students' reflection tools were applied to a set of turtles rather than a single turtle in multiple states, functional aspects of reflection were especially highlighted. For most students, as well as the properties preserved under the transformation, what was emphasised was how every turtle was twinned with an image turtle. It also became clear that the epistemological model of reflection expressed as students re-used command-sets did not the match the model expected by the researcher. Rather than applying the same procedure to all turtles, a common practice in both MTG systems was to change the command set, in systematic ways, according to the position of the turtle. This suggested a view of function as a set of conditional rules not a single general rule.

7.4.3 Comparing across the MTG and DEG systems

Despite obvious differences, mathematical meanings that evolved in the DEG and MTG systems had the following features in common: they could be associated with functional approaches to reflection and both intrafigural and interfigural modes of analysis. There were also some similarities in the processes through which meanings evolved: the support built-in to the microworlds helped students to see and investigate the generality behind geometrical figures, while also allowing them to find ways of expressing this generality in action not formalisation.

7.4.3.1 Connecting the general and the particular

As described in Chapter 3 (*§3.3.2*), geometry comprises figures that are material, visible entities drawn on some surface and, at the same time, theoretical objects that form part of a system with its own axioms, rules of transformation, elements and problems. The goal of constructing microworld tools to represent reflection functions was devised so that students might connect with the dual nature of figures. Although not all students achieved the intended goal, in working towards it they did seem to be building an appreciation of the how particular 'drawings' on screen could be seen as representatives of general construction procedures.

In the DEG systems, the dragging facility played an important role in supporting this appreciation – illustrating to students that they needed to attend not only to what a figure produced on screen looked like, but also to how it behaved when it was moved. Dragging also emphasised the arbitrary nature of particular examples – it was so easy to manipulate points that it made sense to find a general method to position an image point that could be used for any location of the pre-image point or orientation of the axis of symmetry. The general methods students evolved were expressed in two ways: through construction tools, in which case, dragging served as a means of validation; or physically, dragging the points into position to set up particular relationships. Hence, the DEG drag feature that illustrated the difference between drawings and constructions, also provided a way of avoiding engaging in the process of formalisation necessary to construct rather than draw.

In the MTG systems, variables were introduced to illustrate how figures could be defined as general rather than particular cases. As it turned out, it was the structure expressed in the symbolic code used to produce a particular case on screen, which carried the sense of generality for most students – whether or not they managed to incorporate variables into this code. By looking for generality in the symbolic code associated with a particular case, it has been argued that, during their construction activities, the MTG students were building generic examples (see, Mason & Pimm, 1988; Balacheff, 1988), in which, as they attended to particular visual traces on screen, they also considered the general properties they represented.

The interactions possible using the expressive means of both microworlds therefore appeared to support students in making connections between general and the particular. One factor that seemed to be important in encouraging such connections was that, in this study, students had some control over the examples they worked with. This is not usual in paper-and-pencil activities, nor was it a feature of the majority of Logo-based microworlds described in *§3.3.3.4*, in which students worked with computer-generated examples. It may be that it is the lack of just this control that leads students to focus on specific features of the tasks rather than the relational invariants of the transformation.

7.4.3.2 Connecting action and formalisation

One of the reasons for selecting to incorporate microworlds into the learning systems was to support students in formalising as well as identifying the relationships that define the reflection transformation. However, although most students did seem to be thinking about the general properties of the images created during microworld interaction, some pairs in all four learning systems expressed generality in action rather then encoding it in the formal 'text' of the microworld kernel. In DEG systems, this involved soft-constructing and in the MTG systems manually changing values rather than representing them as variables.

In the case of the soft-constructions, when generality was expressed through students' physical manipulations of screen objects, no trace of the action appeared in the text associated with the construction. In contrast, in the MTG microworld, to be enacted turtle movements had to be expressed symbolically. Furthermore, students' manipulations of the MTG symbolic expressions closely modelled the working of the variables they replaced, but this was not necessarily the case with DEG dragging activities. This meant that, while all MTG students attended to both visual and symbolic aspects of their computer constructions, some of the DEG students concentrated only on visual representations. In the DEG systems, the students who did not manage to built a robust reflection construction also tended not to extend their descriptions of reflection in the final interview, while there was no obvious relationship between expression of generality and final-interview descriptions of reflection in the MTG systems. This finding suggests that, when the gap between action and formalisation is not narrowed in computer interaction and no perceptual-theoretical synthesis occurs, the internal resources students' use to make sense of reflection may not correspond to the figural-concepts that Fischbein (1993) argued were necessary in developing understandings of geometry (as described in *§3.3.2*). The implication is that microworlds designed for geometry learning should encourage interaction with both symbolic and visual representations of geometrical constructions.

7.4.3.3 Connections to extra-mathematical activity

In both the DEG and MTG learning systems, students connected the geometrical relationships defined in computational objects with physical movement, but the MTG students were more likely than those who worked with the DEG microworld also to connect what happened on screen with their sense of themselves as people with intentions, goals and desires and to build their own stories to account for the turtles' activities. Making this second kind of connection appeared to encourage feelings of ownership over the tasks and the computational objects and processes involved in them. Allying mathematical activities of turtles with imaginary social practice (such as friendship and romances between screen turtles) also appeared to be a way of giving more meaning to interfigural considerations and shows how, under the right circumstances, students have little difficulty in suspending reality and entering into virtual worlds.

7.4.4 The impact of the instructional approaches

It was found that instructional approach also influenced the trajectories of meaning-making that evolved within learning systems, although rather less than microworld interactions. Specifically, in the DEG-FI system, all students attempted to model the process of reflection using the equal perpendicular property, while a variety of strategies were observed in the DEG-FO system; and in the MTG-FI system, some students began to connect to the notion of geometrical objects as turtle-sets in a way not observed in the MTG-FO system.

These two findings imply some association between instructional approach and microworld – the same variety of methods for modelling reflecting was observed in both MTG systems and connections to the notion of geometrical objects as point sets did not occur in either DEG system. This association may have resulted from differences between the microworlds that themselves can be interpreted in terms of the filling-inwards and filling-outwards models for instruction.

Cabri-géomètre, the software that provided the basis for the DEG microworld, was designed to bring students in touch with a model as close as possible to traditional Euclidean geometry (Laborde & Laborde, 1995). It began, as FI instructional approaches, with the idea of presenting "formal crystallised expert mathematical knowledge" to learners. Turtle geometry, in contrast, "started with the goal of fitting children" (Papert, 1980; p53) and the hope that students would bring what they know about their own bodies and their movements to bear as they learn a formal geometry. In common with the FO instructional approach, the starting point for turtle geometry is hence what is experientially real for the learner.

With these differences between the software in mind, one way of interpreting the first finding associated with instructional approaches is that when a FI approach is used in concert with a FI-based microworld kernel, what could result was the emphasising of one particular, institutionally valued way of thinking about a reflection. The cost may have been the suppression of student invention – precisely the criticism aimed at filling-inwards instructional approaches by constructivist mathematics educators (see, for example, Steffe, 1996; Confrey, 1998; Cobb, Perlwitz & Underwood-Gregg, 1998). Using a FI instructional approach with a FO-orientated microworld kernel did not seem to be associated with the privileging of a particular reflection construction – whether or not students were introduced to general models of reflection before computer interaction, the MTG the models they constructed were firmly related to a turtle-geometry view of reflection and differed quite markedly from the traditional Euclidean geometry definition.

The second of the findings associated with instructional approach related to the expression of knowledge not obviously embedded in the microworld – the view of geometrical objects as point/turtle sets. Among those students' who used the FO-orientated MTG microworld, only those working according to the FI instructional approach connected to this view. The students' in the MTG-FO system did not abstract this from their models – a finding that corresponds to the criticism that student-built models do not necessarily provide the basis for the abstraction of ideas not in mind during model construction (see *§2.4.2*).

It might be tempting to conclude that a FI microworld is best accompanied by a FO instructional approach – and vice versa – but this is too simplistic an interpretation. Students did not connect with the knowledge about point-sets in the DEG system whether or not it was explicitly introduced – a finding that may reflect epistemological differences between the representation of points and turtles in the respective microworlds.

The results of this study suggest that the efficacy of combinations of FI or FO-orientated expressive means and FI or FO instructional approaches is likely also to depend on the specific learning objectives associated with the learning systems. Students in all four systems appeared to extend their knowledge of reflection, but each system had its own particular characteristics.

The DEG-FI system was the only one in which all students referred to the reflection construction traditionally emphasised in school texts, suggesting this system-type might be the most efficient in steering of students to some predetermined set of responses. The greatest variety of reflection constructions were built by a pair in the DEG-FO system, suggesting that this system-type can offer opportunities for students to explore equivalent expressions of the same geometry construction. In the two MTG systems, all students invented and explored their own models of reflection. These models were rather different than the traditional school model, especially in that the perpendicular relationship of reflection was not featured (although some students clearly 'knew' of this property). The MTG-FI was the only system in which students connected to the notion of geometrical objects as point-sets, suggesting that connection to this particular abstraction may be facilitated by a system in which students are encouraged to connect the behaviours of geometrical agents in mathematical systems to with those of human, or at least animate, agents in social systems.

7.4.5 Microworlds and mathematics learning

All four learning systems were associated with evolutions in students' meaning for mathematical knowledge, which occurred while they built computational models of reflection. This general finding provides support for the constructionist thesis that engagement in the building of some external, shareable and personally meaningful product (Papert, 1991) is conducive to mathematics learning. Computational microworlds have a special quality that helps in the construction of mathematical products: the feedback they provide enables a process of learning not unlike the iterative design process adopted in this study. Learners make use of the external resources of the learning system to build the required product, successively observing, analysing and modifying its functioning until the desired effect is achieved.

Constructionism also adds a new dimension to the sociocultural perspective on mediation. In the sociocultural view, emphasis is on the appropriation of the forms of mediation made possible by the use of cultural artefacts (see *§2.2.1* and *§2.2.2*). In the constructionist perspective, the learner becomes involved in creating, as well as appropriating, artefacts that become part of the 'culture' of a learning system. For example, the variable procedure that produced the symmetrical person, Simon, built in the MTG-FI system had an important role in mediating students' meanings for reflection, provided a means of expressing reflective symmetry and served as a means of communicating about the relational invariants of the transformation. Learners' productions can hence have a way of affording and constraining their activities, and the activities of others involved in the learning system, in a similar way to the tools embedded in the system by the designer.

Associating learning with the construction of new tools for expressing knowledge also provides a way of reconciling the constructivist notion of reinvention with the sociocultural view of appropriation. The work of a pair in the DEG-FO systems, for instance, illustrates how, by appropriating the construction tools of the DEG kernel, they were able to (re)invent five different models for reflection.

The role for both appropriation and invention of expressive means in learning systems incorporating microworld interactions suggests a re-examination of the notion of the zone of proximal development. In sociocultural theories of development, this is described as the conceptual site for learning, where knowledge is internalised from the social to the individual plane (see *§2.2.3*). The microworld activities of the four learning systems were designed with this zone in mind, providing access to intended knowledge by embedding it into tools and tasks. However, as the learning systems developed their own identities, the zone of proximal development was further extended by the learners themselves. This was most evident in the MTG systems, where students created imaginary situations, governed by the 'rules' of reflection, as they worked on the task demands. Playing with the objects of the microworld in this way involved detaching from some aspects of the immediate situation and connecting them to new meanings. The implication is that learning systems for mathematics should attempt to capitalise on learners capacity to create new mathematical meanings through play.

7.5 Limitations of the study and suggestions for future research

An assumption underlying the research described in this thesis is that learning occurs in complex, self-organising, interacting systems and the findings have illustrated how learners are sensitive to small changes in conditions. One implication of this assumption is that every learning system examined is to some extent unique. Nonetheless, it has been possible, even given the small sample of students involved, to identify some regularities in the practices that evolved across the learning systems and some ways in which the expressive means and instructional approaches differentially shaped and were shaped by these evolving practices. There are, however, a number factors that limit the scope of the research, suggesting that their generalisability should be interpreted with some caution.

The first relates to a general problem of reporting research associated with the iterative design of learning systems. During each iterative cycle, conjectures about students' learning were made, tools and tasks and teaching interventions were designed, then students' interactions formed the basis for new conjectures about learning. Constraints of time dictated the end of the design process. The detailed analysis of the four systems during the comparison phase suggested there were further modifications that might improve the learning systems. Two particular questions that were raised by analysis were: whether the role of circles in transferring equal distances might be more easily appropriated in DEG learning systems if circle-producing tools were given names that signalled their function, say `equal distances`, rather than the geometrical instrument they model; and whether MTG tools to support the expression of conditional relationships might enable more students to formalise their reflection functions.

At a more general level, when designing the learning systems, attention was given first to the microworld tools and tasks and then to the instructional approaches. This ordering may have contributed to the relative impacts of these different forms of knowledge mediation. In addition, general models of reflection were discussed during a paper-and-pencil-mediated teaching episode. Rather different results may have been obtained if the researcher had introduced microworld-mediated general models into the FI systems.

Turning to the participants of the study, the students were all girls who attended the same single-sex school. The personal ways with which they connected with the tasks may have reflected upon the rather particular culture of adolescent girls and/or the ethos of the school they all attended. It would be interesting to ascertain if situated abstractions of essentially the same nature emerge when mixed-sex or boy-only groups work on the same set of activities.

The researcher acted as the teacher during all the microworld activities. This involvement was an integral part of the iterative design process. Another potential area for future research is whether similar results would be obtained if the designer of the microworld tools had not also had the role of teacher. The researcher had a privileged view of the function and the structure of the tools, as well as the abstractions they were intended to model. Other teachers might construct alternative meanings with these same tools and hence use them in rather different ways with their students.

This raises a whole set of other issues. The tools designed for both the DEG and MTG microworlds were designed with learning about the reflection transformation in mind. Both microworlds made available to students means to express reflection in mathematically-consistent ways that did not necessarily match with textbook definitions. It is only if these alternative expressions are to be accorded the same epistemic value as their traditional counterparts that teachers are likely to invest in learning themselves how to use the microworld tools. Moreover, it would make little sense if teachers and students alike were expected to learn a new set of tools for every curriculum content area. Indeed, all the tools designed in this study were also intended for general use within a particular geometrical system – not just for learning about the transformation reflection. A growing body of research is already addressing the potential of microworlds built in dynamic geometry systems[1], however, the findings of this study suggest the adoption of a multiple-turtle approach to geometry represents an area for further study.

[1] Forthcoming editions of two journals, Educational Studies in Mathematics and International Journal of Computers for Mathematical Learning will be dedicated to research on learning aspects of dynamic geometry.

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Appendix 1

Analysis of the paper-and -pencil tests

Paper-and-pencil tests were developed for use in both the design and comparison phases. Analysis of students' responses to the various versions is presented in this appendix.

A1.1 Tests applied during the design phase

Two versions of a paper-and-pencil test were piloted during the design phase. Each version had three question types, the first two of which were the same in both test versions. The tests were different in one respect: the presence or absence of construction lines on the images presented in the multiple-choice questions (the items are presented in *§5.2.1.*

Part 1: Students' written descriptions of reflection and reflective symmetry

In the first part of the test, students produced written descriptions of reflection and reflective symmetry. The category groups (mirrors, behaviour of light, congruency, reversal and division of space) drawn up on the basis of the interviews (*§5.1.1.2*) turned out to cover all relevant references in students' description with the exception that three students mentioned equal distances (although none were clear about what distances should be equal). Table A1.1 below shows the distribution of responses by group using the extended set of categories.

There was no significance difference ($\chi^2 = 1.574$; df = 6; p>0.05) between the distribution of responses according to test version and similar patterns to those identified in the interviews were also found in students' written descriptions. The most frequent references being those to mirrors (made by 60% of students completing the pink test and 57% the blue) and to congruency (58% and 59% for pink and blue tests respectively). These percentages were rather smaller than they had been in the

the interviews (where over 80% of students had made references to mirror and congruency), but this is not surprising given the differences between the spoken and written contexts.

The over-emphasising of intrafigural over interfigural properties was also evident in the distributions of responses shown in Table A1.1, with references to congruency and reversal of orientation more common than mentions of equal distance. Once again, students wrote of dividing space (29% for pink tests and 22% of students who completed the blue) but made little comment about its structure.

	Pink test n = 45 (% in brackets)		**Blue test** n = 46 (% in brackets)	
Mirrors and other reflective surfaces	27	(60%)	26	(57%)
Behaviour of light	3	(7%)	4	(9%)
Congruency	26	(58%)	27	(59%)
Reversal of orientation	14	(31%)	17	(37%)
Division of space	13	(29%)	10	(22%)
Distance	2	(4%)	1	(2%)
No response	4	(9%)	6	(13%)
			$\chi^2 = 1.574$; df = 6; p>0.05 NS	

Table A1.1: Distribution of students' descriptions of reflection according to common references

In Table A1.2, student responses are distributed according to the conceptualisations: *physical process*, *perceived object* or *property* (see, *§5.1.1.2*). A small number of responses fit none of these classifications, while others included more than one conceptualisation. Again, there was no significant difference in the distribution of responses between the students completing the pink and blue tests ($\chi^2 = 2.488$; df = 6; p>0.05). The most common conceptualisation in both groups was that of reflection as perceived object, again supporting the conjecture that within-figure properties of the pre-image and image are the main focus of attention for students.

	Pink test n = 45 (% in brackets)		Blue test n = 46 (% in brackets)	
Physical process	17	(38%)	13	(28%)
Perceived object	21	(47%)	18	(39%)
Property	8	(18%)	13	(28%)
Unclassified	4	(9%)	3	(7%)
No response	4	(9%)	6	(13%)
				$\chi^2 = 2.488$; df = 6; p>0.05 NS

Table A1.2: Distribution of students' descriptions of reflection according conceptualisation

Part 2: Students' drawings

Turning to the students responses to the six construction items, the range of student responses to each question are presented in the Tables A1.3 – A1.8.

For the first four construction items, responses have been classified according to the following criteria: *reflection in given axis* (which indicates that students produced the desired response), *other isometries* (where students performed a different reflection, a glide reflection, translated, or rotated the pre-image), *other* if the image cannot be described by any isometric transformation and *blank* where no answer was given (see, Tables A1.3 – A1.6). A margin of error ±0.2cm was used when judging the congruency of pre-image and image measures. For all four items, the majority of responses fell in the first two categories, indicating that most students constructed images that were congruent to the pre-images (on items 3 and 4 in which the pre-image was placed horizontally or vertically, all images were congruent).

No significant differences in the distribution of responses according to test version were found for any of these four items (item 1: $\chi^2 = 0.76$; df = 2; p>0.05; item 2: $\chi^2 = 7.708$; df = 3; p>0.05; item 3: $\chi^2 = 2.464$; df = 2; p>0.05; item 4: $\chi^2 = 03.96$; df = 2; p>0.05). Despite, not reaching a level of statistical significant, one consistent difference between the two student groups was noted: Students who completed the blue test more frequently constructed images corresponding to isometric transformations other than reflection in the given axis than the students who responded to the pink test.

Given the similarity between items 1 and 2, the drop in correct responses from the first to the second (20% for both groups) was slightly surprising. Perhaps the difference corresponded to a greater inclination to students imagine 3-D situations in connection to second item (where the pre-image resembled as flag) as compared to the first one. This tendency could also have contributed to the decrease in the number of sketches presenting correct 2-D responses when the axis was placed at an 45° angle as compared to the horizontal or vertical.

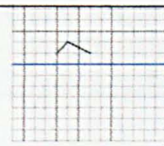
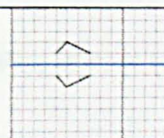
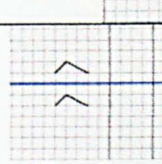
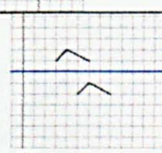
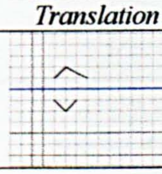
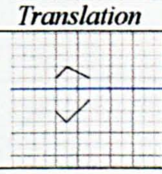
ITEM 1		**Pink test** n = 45 (% in brackets)	**Blue test** n = 46 (% in brackets)
Reflection in given axis		38 (84%)	41 (87%)
Other Isometries	 *Translation*  *Translation*	3 (7%)	3 7%
Other	 	4 (9%)	2 (4%)
Blank		0 (0%)	0 (0%)
$\chi^2 = 0.76$; df = 2; p>0.05 NS			

Table A1.3: Responses to first construction item

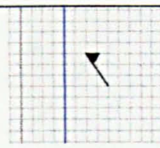
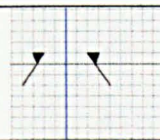
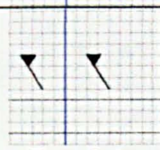
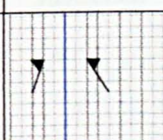
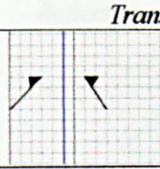
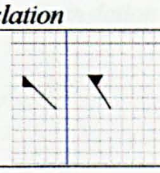
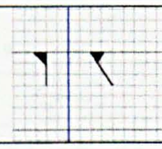
ITEM 2		**Pink test** n = 45 (% in brackets)	**Blue test** n = 46 (% in brackets)
Reflection in given axis		29 (64%)	31 (67%)
Other isometries	 *Translation*	5 (11%)	12 (26%)
Other	   	10 (22%0	3 (7%)
Blank		1 (3%)	0 (0%)
			$\chi^2 = 7.708$; df = 3; p>0.05 NS

Table A1.4: Responses to second construction item

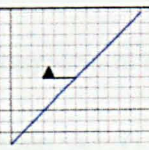
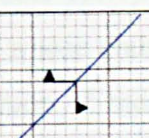
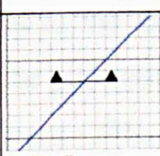
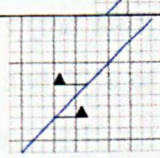
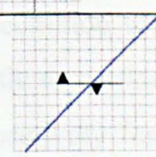
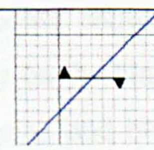
ITEM 3		**Pink test** n = 45 (% in brackets)	**Blue test** n = 46 (% in brackets)
Reflection in given axis		24 (53%)	17 (37%)
Other isometries	 *Reflection*  *Glide reflection*  *Glide reflection*  *Rotation*	21 (47%)	29 (63%)
Blank		0 (0%)	0 (0%)
			$\chi^2 = 2.464$; df = 2; p>0.05 NS

Table A1.5: Responses to third construction item

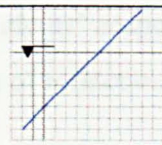
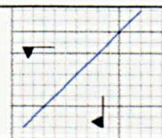
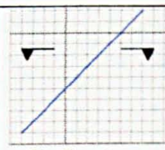
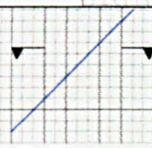
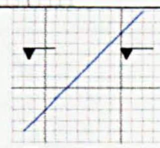
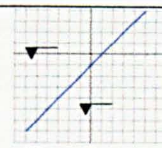
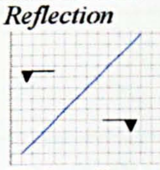
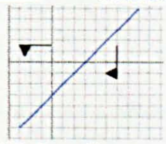
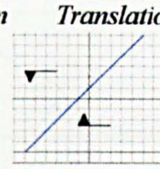
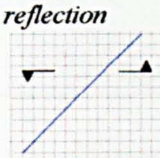
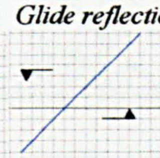
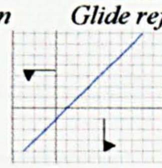
ITEM 4		**Pink test** n = 45 (% in brackets)	**Blue test** n = 46 (% in brackets)
Reflection in given axis		16 (36%)	9 (20%)
Other isometries	    *Reflection* *Reflection* *Translation* *Translation*    *Glide reflection* *Glide reflection* *Glide reflection*    *Rotation* *Rotation* *Rotation*	26 (58%)	34 (74%)
Blank		3 (7%)	3 (7%)
		$\chi^2 = 3.96$; df = 2; $p>0.05$ NS	

Table A1.6: Responses to fourth construction item

Responses to the fifth item (Table A1.7) did not follow the same pattern as the previous four. Given a segment whose ends were not located on intersections points of the underlying grid, the reaction of most students was to draw image-segments either parallel to the axis (the most common response chosen by 53% of students completing the pink test and 50% of those working on the blue) or as parallel to the original segment (29% and 30% respectively for pink and blue). This finding is at odds with the other items, as students more frequently co-ordinated the image with the axis than with the pre-image. One possible explanation relates to the presence of the grid. When items are presented on square paper, it is common to work with lines drawn horizontally, vertically or with diagonal lines through grid points. This may have been why many students had chosen to sketch the image at the same 45° angle as the axis. It may also have been that the axis was chosen as reference in this item because an image-segment parallel to it is closer to the true reflective image than an image segment drawn as parallel to the original segments. In any case, the grid seemed to have mediated in students' responses in ways that emphasised properties not related to the transformation of reflection.

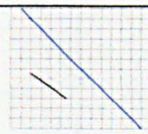
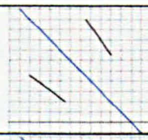
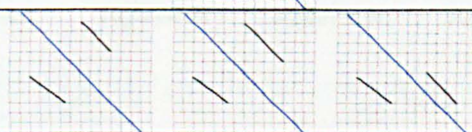
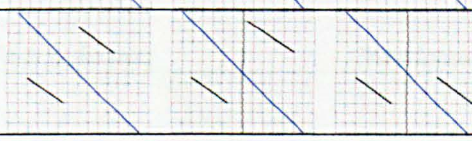
ITEM 5		**Pink test** n = 45 (% in brackets)	**Blue test** n = 46 (% in brackets)
Reflection in given axis		5 (11%)	8 (17%)
Image parallel to axis		24 (53%)	23 (50%)
Image parallel to pre-image		13 (29%)	14 (30%)
Blank		3 (7%)	1 (2%)
$\chi^2 = 0.695$; df = 3; p>0.05 NS			

Table A1.7: Responses to fifth construction item

For the last construction item in which a segment crossed the axis (see, Table A1.8), the majority of students from both groups (56% and 54%) came up with sketches that involved reflecting only one half of the figure (see, Table A1.8). In retrospect, it was felt that the original figure they had been given may have exacerbated any tendency to do this as the smaller segment was almost positioned such that it could be seen as a partial reflection of the longer (and many students either extended or 'bent' the segment). Even given this, the success rates were low, with very few students, especially among those completing the pink test, producing the desired image.

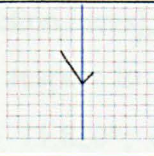
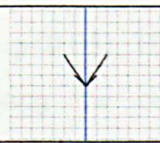
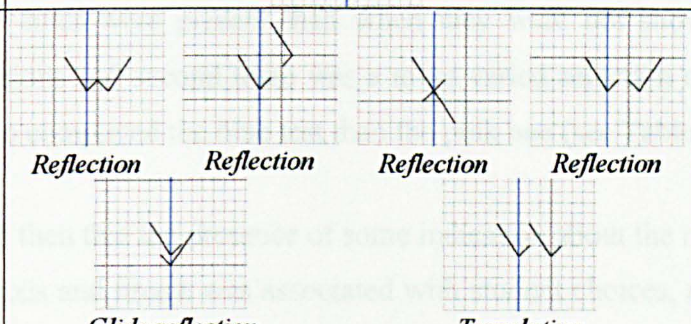
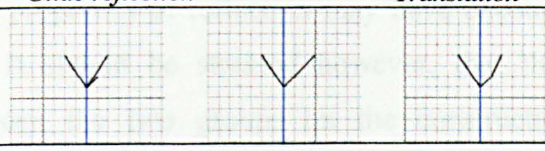
ITEM 6		**Pink test** n = 45 (% in brackets)	**Blue test** n = 46 (% in brackets)
Reflection in given axis		4 (9%)	7 (15%)
Other Isometries	 *Reflection* *Reflection* *Reflection* *Reflection* *Glide reflection* *Translation*	12 (27%)	10 (22%)
Attempts to transform semi-plane		25 (56%)	25 (54%)
Blank		4 (9%)	4 (9%)
$\chi^2 = 0.989$; df = 3; p>0.05 NS			

Table A1.8: Responses to sixth construction item

One last comment can be made about students' attempts to control the distances between images, pre-images and axis. In doing so, were they beginning to focus on interfigural relationships? That is, were they beginning to try to mathematise the relationship between the figures as well as within them, by searching for the relationships by which the image could be co-ordinated with both pre-image and axis? It is difficult from their test responses to be certain, but what seems to characterise the majority of student responses is that distance is controlled to satisfy the following criteria: if a ruler is placed (or line imagined) between pre-image and image (usually, but not always, between a vertex of the pre-image and its corresponding image point) then the point at which the ruler crosses the axis is half-way between pre-image and image. Although this treatment of distances involves two objects, its explicit parts at least are essentially intrafigural.

Part 3: Recognising reflections

While there were no significant differences between the students completing the pink tests as compared to the blue on the parts that were the same (the written descriptions of reflection and the six construction items), significant differences in the distributions of responses were found with respect to both the multiple-choice questions (first multiple-choice question: $\chi^2 = 11.820^*$; df = 4; $p<0.05$, second multiple-choice question: $\chi^2 = 13.052^*$; df = 4; $p<0.05$). There are two main observations. First, the percentage of students choosing the correct response was higher when the construction lines were present than when they were not shown on both multiple choice questions and second there was a more varied selection of options among the students who completed the blue test than the pink one (see Tables A1.9 and A1.10).

It does seem then that the presence of some indication about the relationships between pre-image, axis and image was associated with student choices, and that some sort of external sign of properties of reflection may have encouraged students to make the correct choice. It should be stressed however, that there was a non-significant difference between the two groups on the construction items, where students completing the blue tests had produced images corresponding to isometric transformations other than the given reflection than those working on the pink test. So, it could be one group of students was simply slightly more predisposed to selecting the wrong transformation and this was accentuated in the multiple-choice questions. On the other hand, it could also be that access to the construction lines on the multiple-choice question had also effected the way students had sketched the images on the construction items. It is also notable that a small number of students selected more than one option (even though the question had explicitly asked for one).

	Pink test n = 45 (% in brackets)		Blue test n = 46 (% in brackets)	
Arthur's (image = reflection in vertical axis)	14	(31%)	13	(28%)
Boris's (image = rotation)	2	(4%)	5	(11%)
Carol's (image = reflection in given axis)	28	(62%)	18	(39%)
Dorothy's (image = glide reflection)	0	(0%)	8	(17%)
More than one	1	(2%)	2	(4%)
				$\chi^2 = 11.820^*$; df = 4; $p<0.05$

Table A1.9: Distribution of students' responses to first multiple-choice question

	Pink test n = 45 (% in brackets)		Blue test n = 46 (% in brackets)	
Elaine's (image = reflection in perpendicular bisector of line joining flag poles)	4	(9%)	11	(24%)
Femi's (image = reflection in given axis)	35	(78%)	26	(57%)
Gregory's (image = rotation)	1	(2%)	4	(9%)
Hamble's (image = translation)	1	(2%)	5	(11%)
Isobel's (image = rotation)	0	(0%)	0	(0%)
More than one	4	(9%)	0	(0%)
				$\chi^2 = 13.052^*$; df = 4; $p<0.05$

Table A1.10: Distribution of students' responses to second multiple-choice question

A1.2 Tests applied during the comparison phase

The final test version is described in *§6.1* and the complete set of test items can be found in Appendix 3. The test had five different parts, the results from which are described below.

Part one: Students' written descriptions of reflection and reflective symmetry

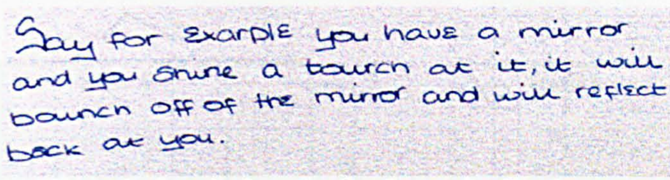
Table A1.11 shows the distribution of student responses according to the most frequently used references and Table A1.12 shows their spread amongst the three different conceptions of *reflection as physical process, perceived object* or *property* (as described in *§5.2.1.2*). Within both coding systems where student's descriptions could be interpreted according to more than one category they received multiple codes.

The distribution of responses presents similar patterns to those identified in the design phase. Reflection was again frequently associated with use of mirrors (66%), and fifteen students (17%) made references to the behaviour of light in relation to mirrors or other reflective surfaces. The results also confirm that it is intrafigural relations which dominate students' written views of reflection and reflection symmetry, with congruency in some form the most popular reference (included in the descriptions of 72% of students). Ten (11%) students wrote of reversal in orientation of the image in relation to the original, while only one of the eighty-eight students explicitly mentioned the distance between the axis (line) of reflection and other elements involved in the transformation. References to space are limited to its divisions into two sides, divided by a line or surface. Such a reference occurred in the descriptions of thirty-two descriptions (36%).

	No. of students **n = 88** **(% in brackets)**	
Mirrors and other reflective surfaces	58	(66%)
Behaviour of light	15	(17%)
Congruency	63	(72%)
Reversal of orientation	13	(15%)
Division of space	32	(36%)
Distance	1	(1%)
No response	3	(3%)

Table A1.11: Distribution of students' descriptions of reflection according to common references

The conceptualisation that emerged most frequently within this student sample was reflection as a perceived object (36%), although there was a more even spread of responses amongst the three categories than had been the case in among the pilot responses. This suggests there was more awareness on reflection as a process and as a property in this sample of students as compared to the previous group. Nonetheless, it was still the case that students described the products of transformation more frequently than the process (the property that students mention is between two parts of the final configuration in which pre-image, image and axis are all present). And as in the design phase, where reflection was described as a process, the process was described in physical rather than mathematical terms as the example in Figure A1.1 illustrates:



A1.1 Kirsten's description of reflection and reflective symmety

	No. of students n = 88 (% in brackets)	
Physical process	31	(35%)
Perceived object	34	(39%)
Property	32	(36%)
Unclassified	6	(7%)
No response	3	(3%)

Table A1.12: Distribution of students' descriptions of reflection according to conceptualisation

Part 2: Drawing reflections

Moving on to students' responses to the reflection items which are presented in Tables A1.13, A1.14 and A1.15 below, once again the results clearly indicate that students are most likely to sketch correct 2-D responses when the axis of symmetry has either a horizontal or vertical orientation (item 1: 81%; item 2: 65%; item 5: 63%)[1], with a drop in the number of correct responses observed on the items more

[1] In judging responses, a margin of error of ±0.2cm on all distances and of ±5° on angles was allowed.

suggestive of 3-D situations (compare item 1 to item 5). When the axis was inclined, the percentage of correct responses dropped and correct 2-D representations were rarely produced when the axis was inclined at an angle to the horizontal that was not a multiple of 45° (in item 6, only 6% of responses – or five students – represented reflections in the given axis). Taken together with the low percentage of correct 2-D responses for the item presented a segment whose endpoints were not placed on the intersection points of the grid (item 4, Table A1.15), this provides further evidence that the presence of the squares mediates students' responses in ways that sometimes help produce correct images by using strategies that do not generalise well to all possible configurations of the plane.

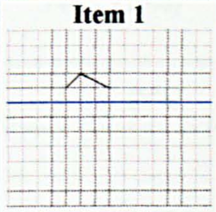
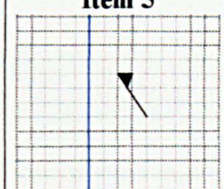
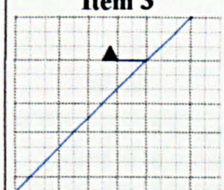
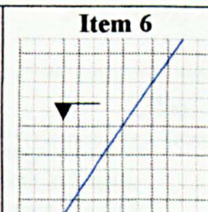
Response type	**Item 1**  **No. of students** n = 88 (% in brackets)	**Item 5**  **No. of students** n = 88 (% in brackets)	**Item 3**  **No. of students** n = 88 (% in brackets)	**Item 6**  **No. of students** n = 88 (% in brackets)
Reflection in given axis	71 (81%)	55 (63%)	38 (43%)	5 (6%)
Other isometries	8 (9%)	23 (26%)	40 (45%)	61 (69%)
Non-congruent images	9 (10%)	13 (15%)	10 (11%)	15 (17%)
Blank	0 (0%)	2 (2%)	0 (0%)	7 (8%)

Table A1.13: Students' responses to items involving axes or pre-images with horizontal or vertical orientation

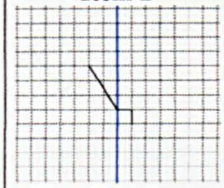
Response type	Item 2  **No. of students** n = 88 (% in brackets)
Reflection in given axis	57 (65%)
Other isometries	10 (11%)
Reflection of semi-plane	3 (3%)
Non-congruent images	14 (16%)
Blank	4 (5%)

Table A1.14: Students' responses to items with pre-images crossing axis

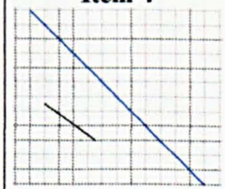
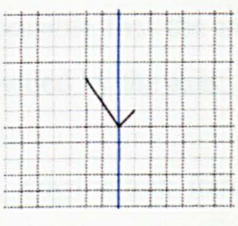
	Item 4  **No. of students** n = 88 (% in brackets)
Reflection in given axis	21 (24%)
Image parallel to axis*	40 (45%)
Image parallel to pre-image**	19 (22%)
Others	5 (6%)
Blank	3 (3%)
* 30 of the 40 image segments were judged to be congruent in length. ** 10 of the 19 image segments were judged to be congruent in length.	

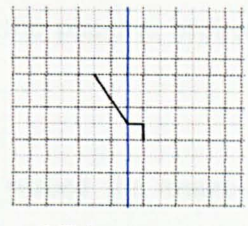
Table A1.15: Students' responses to item with segment not on grid

It is interesting to compare the differences between the results to item 2 in this test version and item 6 in the pilot. In both a figure crosses a vertical axis of symmetry, but students were much more likely to construct the correct 2-D for one than for the other (see Figure A1.2). There are a number of differences between the two figures that might have contributed to the difference in constructions. In particular, it is easier to imagine breaking down into segments the figure used in the final version than the one used in the pilot and the pilot version too seems more suggestive of a 3-D perspective. If students were more likely to treat the pilot version as a whole object, then it may have been for this reason they appeared to operate with the space on only one side of the axis. Another interpretation was that the smaller segment one side of the axis already represented part of reflective image of the other side in the pilot version. Certainly the majority of students working with the item in the final test-version did not treat the axis as a "single-sided" mirror, but as the line divided an as yet to be completed figure into congruent parts with opposite orientation, a basically intrafigural analysis.



Correct 2-D image: 14%

Image of one side only: 55%



Correct 2-D image: 65%

Image of one side only: 3%

Figure A1.2: Two figures that cross the axis of reflection

Although in their descriptions of their knowledge about reflection, references to distances from the axis were extremely rare, as Table A1.16 illustrates, a large majority of attempted to take account of distances in their drawings. On most questions the percentage of students controlling distance is well over 70%. One possible explanation for this difference in action and expression is that, when producing these drawings, students are not only considering the highlighted figure as whole object, but tend to think that reflection operates on whole "sides", that is the whole of the space visible on one side of the axis is the same of the whole of the space visible on the other (Miriam's description in Figure A1.3 corresponds to such a view).

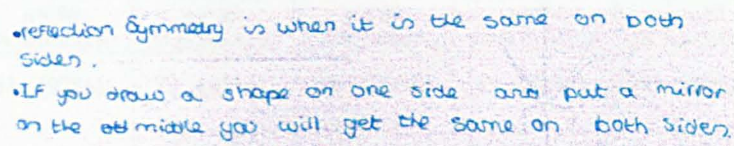
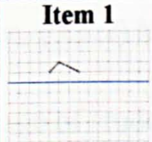
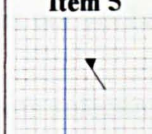
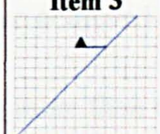
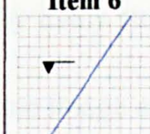
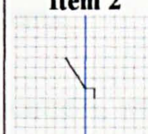
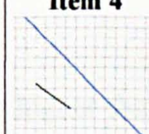


Figure A1.3: Miriam's description of reflection and reflective symmetry

Miriam connects her view to the use of mirrors, but folding could also be connected to such a view. Ironically, in this view which takes account of the whole space involved in the transformation and may therefore be considered to have some connection with the notion of mappings of the whole plane, the defining feature of the isometries, the preservation of distances between points, remains at an implicit level. Actually, one explanation as to why the term distance is virtually absent from students' description of reflection is precisely because of this view of reflection as applied to sides: Since sides are adjacent to the axis of reflection, the distance is zero, which might be interpreted as no distance at all. The implication of this is that the idea of transformation as a mapping of one figure onto another is neither necessarily more nor less accessible that the mapping of plane, where the plane is broken down into two parts divided by the axis.

In the final construction item, rather fewer students placed the image-figure at an equal distance to the axis as the pre-image-figure as compared to the other items (53% asserted some control over distance). This was because of those who constructed images correspond to different isometric transformations the most common was a glide reflection. In this glide reflection, the image was consistent with a reflection in a vertical axis and then a horizontal translation of one square to the left (29 students came up with this figure). It seems very likely that this was to do with the grid on which the pre-image was presented, in which a reflection in the vertical axis would result in an image figure that was not entirely contained by the grid – another example of its mediational role.

	Item 1  **No. of students** n = 88 (% in brackets)	**Item 5**  **No. of students** n = 88 (% in brackets)	**Item 3**  **No. of students** n = 88 (% in brackets)	**Item 6**  **No. of students** n = 88 (% in brackets)	**Item 2**  **No. of students** n = 88 (% in brackets)	**Item 4**  **No. of students** n = 88 (% in brackets)
Distance from axis conserved*	81 (92%)	72 (82%)	84 (95%)	47 (53%)	82 (93%)	65 (74%)
Distance not comserved	7 (8%)	12 (14%)	4 (5%)	34 (39%)	2 (2%)	20 (23%)
Blank	0 (0%)	4 (5%)	0 (0%)	7 (8%)	4 (5%)	3 (3%)

* Conservation of distance signifies the presence of an equal distances between one pre-image-image point pair and the axis or of clear indication of some other method to attempt to place image (as whole object) at an equal distance from the axis as the pre-image.

Table A1.16: Students' treatment of distance

Part 3: Recognising reflections

Tables A1.17 and A1.18 present the distribution of students' choices on the two recognition questions. In both cases the correct 2-D representation was the most popular choice, although it was selected by less than half the students in both cases, (43% for the vertical pre-image figure and 49% when pre-figure was presented on a slant). There was a difference between these items as presented in the pilot tests and the final version, specifically the students were presented with both construction lines and the measures of distances and, where appropriate, angles showing how the lines had been drawn. Values of measures had not been presented on either of the pilot test versions (one had included construction lines and one had no information). Apart from the most common response, students' tended to make different choices in the presence of these measures than when they were not available. It is possible that, for many students, the visual signs when presented alone served a qualitative role, but when accompanied by numbers this changes and quantitative relationships are emphasised. Perhaps because the quantitative measures always signified equality, students were more tempted to choose as correct representations items that qualitative assessments alone would have rejected. For example, Isobel's answer was judged as correct by none of the students completing the pilot versions of the test, while seven students in this sample thought it was correct. This qualitative-quantitative divide suggests the harmony between the visual/theoretical dialectic characteristic of figural concepts (see, *§3.3.2*) has not yet been achieved by many students.

The percentage of students selecting more than one image as correct, was higher than had been the case in the pilot version. This reflects a more ambiguous wording in the final versions, where students were asked to indicate which choices were correct, rather than choosing just one. The wording on the final version quite possibly elicited a more accurate indication of the percentage who believed that a particular pre-image might have more than one image.

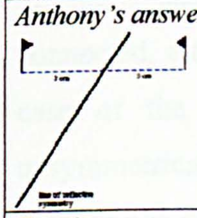
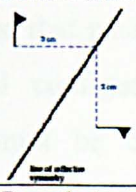
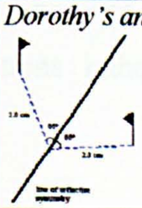
Choice	No. of students n = 88 (% in brackets)	Choice	No. of students n = 88 (% in brackets)
Anthony's answer 	13 (15%) *9 (10%)*	*Boris's answer* 	17 (19%) *8 (9%)*
Carol's answer 	38 (43%) *12 (14%)*	*Dorothy's answer* 	3 (3%) *3 (3%)*
More than one choice	14 (16%)	No choice	3 (3%)
Numbers in italics show how many times choice was selected as one of various possibilities			

Table A1.16: Distribution of students' responses to first recognition multiple-choice question

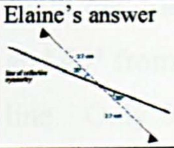
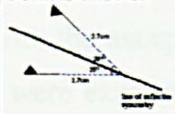
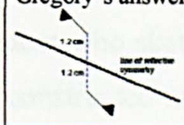
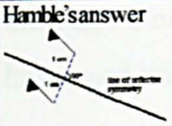
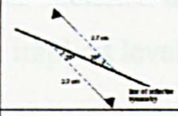
Choice	No. of students n = 88 (% in brackets)	Choice	No. of students n = 88 (% in brackets)	Choice	No. of students n = 88 (% in brackets)
Elaine's answer 	5 (6%) *4 (5%)*	Femi's answer 	43 (49%) *5 (6%)*	Gregory's answer 	7 (8%) *7 (8%)*
Hamble's answer 	11 (13%) *4 (5%)*	Isobel's answer 	3 (3%) *4 (5%)*		
More than one choice	11 (13%)	No choice	8 (9%)		
Numbers in italics show how many times choice was selected as one of various possibilities					

Table A1.18: Distribution of students' responses to second recognition multiple-choice question

Part 4: Identifying properties of reflection

In the fourth part of the paper-and-pencil test students were asked to respond to four more multiple-choice questions. A complete analysis of the responses to this part can be found in *§6.1.1.*

Part 5: Constructing an axis of symmetry

Four main strategies were identified amongst those used by students to construct the missing axis of reflection: *perceptual*, when its position was drawn "by eye"; *part theoretical/part perceptual*, measuring the distance between the two points and placing a midpoint half-way between them, then drawing by eye a line passing through this midpoint; *theoretical*, in which the midpoint was constructed and then the axis, passing through this midpoint, placed at 90° to the segment which joined the pre-image and image points; and *joining* where the pre-image and image points were connected, either with a segment or a line that passed through them. In the second case of the joining strategy, the final configuration is not strictly speaking unsymmetrical, but the two points cannot be considered to be a pre-image-point/image point pair. This strategy could be used as evidence to suggest students' goals related to symmetrical configurations rather than functional relationships associated elements of sets.

Table A1.19 presents the distribution of student responses according to strategy. As can be seen, the strategy of controlling one distance then drawing the axis by eye was the most common strategy (used by 63% of students). The accuracy of the sketched axis varied considerably, but in the all the responses involving the axis was inclined (with the orientation representing a clockwise rotation or between approximately 10° and 40° from the vertical), with the exception of one student who sketched a vertical line. Only 3 students (3%) were explicit about having constructed an angle of 90°, although a number of students sketched lines close enough to suggest that they might have used this property at an implicit level.

	No. of students n = 88 (% in brackets)	
Perceptual	6	(7%)
Part theoretical -part perceptual	55	(63%)
Theoretical	3	(3%)
Joining	17	(19%)
Other or blank	7	(8%)

Table A1.19: Distribution of students' strategies for producing a missing axis of reflection

Appendix 2

Definitions of microworld tools

This appendix contains the formal descriptions of the DEG and MTG tools.

A2.1 The DEG tools

Five DEG construction tools were added as macros to the construction menu of Cabri-géomètre I (version 2.1). These were designed as follows:

```
Compass(2)
Input: 2 points, A and B
P3 (invisible): midpoint A B
P4 (invisible): symmetrical point of A in P3
Output: A circle with centre A and circumference passing
through P4.

Compass(3)
Input: 3 points, A, B and C
P4 (invisible): midpoint B C
P5 (invisible): symmetrical point of A in P4
Output: A circle with centre C and circumference passing
through P5.

Angle carry(3)
Input: 3 points, A, B and C
L1 (invisible): line through B and C
P4 (invisible): symmetrical point of A in L1
Output: A line through B and P4

Angle carry(4)
Input: 4 points, A, B, C and D
P5 (invisible): midpoint of C and D
Output: line made using angle carry (5) on points A, B, C, P5
and D

Angle carry(5)
Input: 5 points, A, B, C, D and E
P6 (invisible): midpoint of E and A
P7 (invisible): symmetrical point of B in P6
P8 (invisible): midpoint of E and C
P9 (invisible): symmetrical point of B in P8
L1 (invisible): angle bisector of P9, E and D
P10 (invisible): symmetrical point of P7 in L1
Output: A line through E and P10
```

A2.1 The MTG tools and interface

Figure A2.1 presents the MTG microworld with all screen artefacts shown and the descriptions of the tool for communicating with turtles can be seen in Figure A2.2.

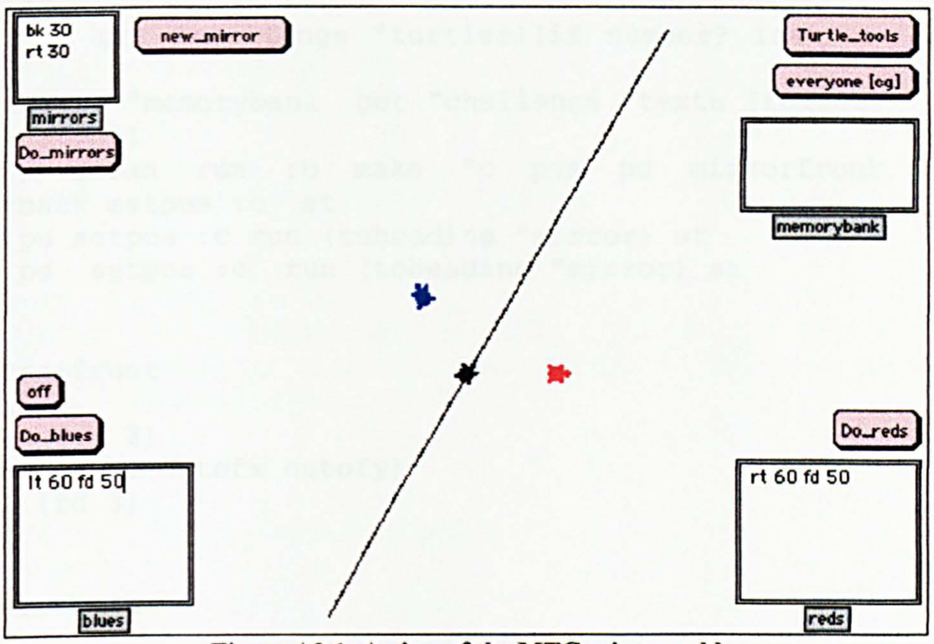


Figure A2.1: A view of the MTG microworld

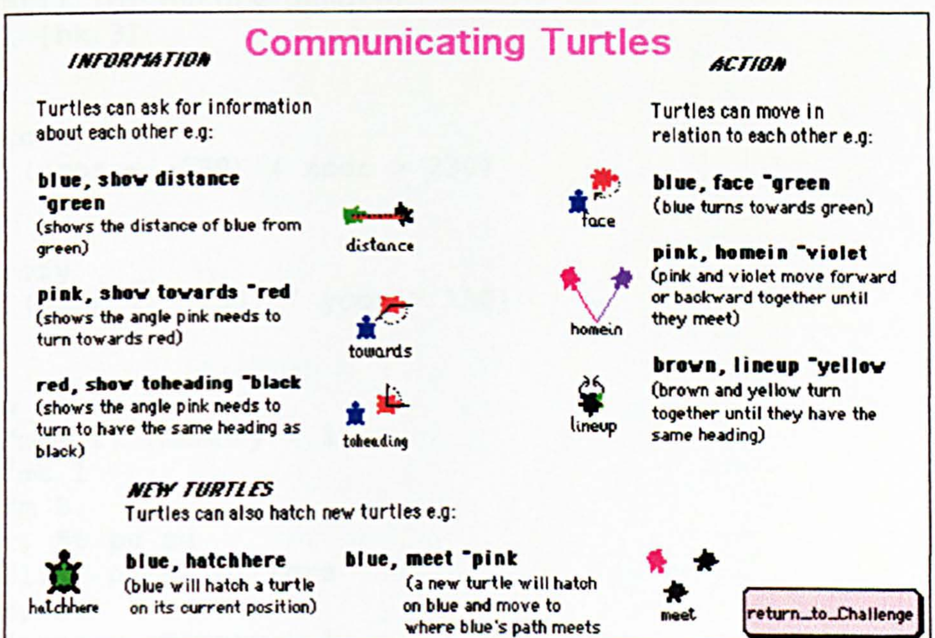


Figure A2.2: The turtle communication tools

The following listing shows the programming behind this artefacts and tools.

```
;;; PROCEDURES FOR SETTING THINGS UP

to mirrorline :b
local "c
dolist [i get "challenge "turtles][if number? last :i [remove
:i]]
if  member? "memorybank  get "challenge "texts [forget]
everyone [ht]
mirror, clean run :b make "c pos pd mirrorfront bk 10
mirrorback setpos :c  st
blue, pu setpos :c run (toheading "mirror) st
red, pu  setpos :c  run (toheading "mirror) st
end

to mirrorfront
mirror,
forever [fd 3]
waituntil [or outofx outofy]
cancel [fd 3]
end

to mirrorback
mirror,
forever [bk 3]
waituntil [or outofy outofx]
cancel [bk 3]
end

to outofx
op or (xcor < -230) ( xcor > 230)
end

to outofy
op or (ycor <  -150) ( ycor > 150)
end

to new_mirror
make "memory :memory + 1
make "mt 1
make "m 0
mirror, ht pu cg
mirrorline parse mirrors
mirror, st
make "memory :memory - 1
end
```

;;; **PROCEDURES FOR RECORDING AND USING HISTORIES**

```
to startup
on
freeze "blues
freeze "reds
freeze "mirrors
;;;turtlesown "orientation
everyone [setorientation 1]
make "mt 0
red, setpensize 2
blue, setpensize 2
make "turtlist get "challenge "turtles
end

to llogit :list
if :memory = 0 [ifelse who = "blue [blues, pr :list][ifelse
who = "red [reds, pr :list][ ( se [There is no] who [box where
I can put] :list )]]]
end

to off
set "button3 "rule [launch [on]]
make "memory 0
end

to on
set "button3 "rule [launch [off]]
make "memory 1
end

to do_blues
copy
blue, run parse clipboard
end

to do_reds
copy
red, run parse clipboard
end

to do_mirrors
copy
mirror, run parse clipboard
make "c pos ht pd   mirrorfront bk 10 mirrorback setpos :c  st
end
```

```
;;; NEW TOOLS and their use

;;; reminder of the tools

to Turtle_tools
getpage "Turtle_tool_kit
end

to return_to_Challenge
getpage "Challenge
end

;;; turn turtle over

to flip
setorientation orientation * -1
end

;;; swop lefts and rights

to swop :list
flip run :list flip
end

;;; hatch a new turtle

to hatchhere
local [a name]
make "a who
ifelse number? last :a [make "name word bl :a (count
turtlenames :a)][make "name word :a (count turtlenames :a)]
newturtle :name setsize 20 setc ask :a [color] setpos ask :a
[pos] seth ask :a [heading] setorientation 1
end

to turtlenames :t
local [sublist]
make "sublist []
dolist [i get "challenge "turtles ] [if (or :i = :t (bl :i)
= :t :i = (bl :t) (bl :i) = (bl :t))[ make "sublist se
:sublist :i]]
op :sublist
end
```

;;; create new mirror turtle where red or blue is pointing

```
to hatchmirror
local [a :x]
make "x who
make "a "mirror
make "mt :mt + 1
newturtle word "mirror :mt setsize 20  setorientation 1 setpos
ask :a [pos] seth ask :a [heading]
make "memory :memory + 1
keepon :x word "mirror :mt
make "memory :memory + 1
tto :x
end

to keepon :a :b
if (ask :a [last towards :b]) = 0 [stop]
ask :b [fd whichway :a :b]
keepon :a :b
end

to whichway :a :b
local "f
make "f ask :a [last towards :b]
ask :b [ht pu fd 1]
ifelse (abs ((ask :a [last towards :b]) - :f)) = 1 [ifelse
(ask :a [last towards :b]) > :f [make "f -10][make "f
10]][ifelse (ask :a [last towards :b]) > :f [make "f -
1.5][make "f 1.5]]
ask :b   [bk 1 st]
op :f
end
```

;;; ouput the angle to give one turtle the same heading as another

```
to toheading :x
ifelse (diff :x) > 180 [op list "rt 360 - diff :x][ifelse
(diff :x) > 0 [op list "lt diff :x ][ifelse (abs (diff :x)) >
180 [op list "lt 360 - abs diff :x][op list "rt abs diff :x
]]]
end

to diff :b
;;;show  (ask who  [heading]) -  (ask :b [heading])
op  (ask who  [heading]) -  (ask :b [heading])
end
```

;;; move turtle to where it "intersects" with another

```
to intersect :b
local [a f]
make "a who
if (ask :a [towards :b]) = 0 [stop]
ask :b [fd whichway :a :b]
intersect :b
end
```

;;; make two turtles move together to same point/angle

;;;(1)when they are symmetrical

```
to homein :b
make "memory :memory + 1
local [f a ]
make "dist 0
make "a who
make "f way :a :b
forever [ make "dist :dist + 1 ask se :a :b  [fd :f]]
waituntil[ touching?  :a :b]
cancel  [ make "dist :dist + 1 ask se :a :b  [fd :f]]
make "memory :memory - 1
ifelse :f < 0 [tto :a logit se "bk :dist tto :b logit se "bk
:dist][tto :a logit se "fd :dist tto :b logit se "fd :dist]
end

to touching? :a :b
op ask :a [(distance :b) < 1]
end

to lineup :b
make "turn 0
make "memory :memory + 1
local "a
make "a who
forever [ask :a [rt 1] ask :b [lt 1] make "turn :turn + 1]
waituntil [sameheading? :a :b]
cancel [ask :a [rt 1] ask :b [lt 1] make "turn :turn + 1]
make "memory :memory - 1
tto :a logit se "rt :turn tto :b logit se "lt :turn
end

to way :a :b
local "f
make "f ask :a [distance :b]
ask :a [ht pu fd 10]
ifelse (ask :a [distance :b]) > :f [make "f -1][make "f 1]
ask :a [bk 10  pd st]
op :f
end
```

```
;;; (2)by checking who gets them nearer

to meet1 :b :c
local [a y1 y2]
make "a who
make "y1 ask :a [pos]
make "y2 ask :b [pos]
forever [trial :a :b]
waituntil [touching? :a :b]
cancel [trial :a :b]
ask :a [hatchhere :c]
ask :a [setpos :y1]
ask :b [setpos :y2]
end

to trial :a :b
local [cd1 cd2]
make "cd1 checkdist :a :b
make "cd2 checkdist :b :a
ifelse (:cd1) > (:cd2) [ask :a [fd 1 ]] [ifelse :cd1 < :cd2
[ask :b [fd 1 ]][ask se :a :b [fd 1]]]
;;;show ask :a [distance :b]
end

to checkdist :a :b
local [d]
ask :a [fd 1]
make "d (ask :a [distance :b])
ask :a [bk 1]
op (ask :a [distance :b]) - :d
end
```

```
;;; by calculating distance to meeting point and moving in
steps of 20

to meet :b
local "a
make "a who
meet3 :a :b
remove first get "challenge "turtles
tto first get "challenge "turtles
end

to meet3 :a :b
local [d a1 b1 t1 t2 w1 w2]
make "d  ask :a [distance :b]
make "t1 ask :a [turnsize :b]
make "t2 ask :b [turnsize :a]
ask :a [hatchhere]
make "a1 first get "challenge "turtles
ask :b [hatchhere]
make "b1 first get "challenge "turtles
make "w1 way :a :b
make "w2 way :b :a
ifelse cond1 :t1 :t2 [act1 :a1 :b1 :d last :t1 last :t2 :w1
:w2] [ifelse cond2 :t1 :t2 [ act2 :a1 :b1 :d last :t1 last :t2
:w1 :w2] [ifelse cond3 :t1 :t2 [  act3 :a1 :b1 :d last :t1
last :t2 :w1 :w2] [ifelse cond4 :t1 :t2 [act4 :a1 :b1 :d last
:t1 last :t2 :w1 :w2][ifelse cond5 :t1 :t2 [ act5 :a1 :b1 :d
last :t1 last :t2 :w1 :w2][ifelse cond6 :t1 :t2 [act6 :a1 :b1
:d last :t1 last :t2 :w1 :w2] [ifelse cond7 :t1 :t2 [ act7 :a1
:b1 :d (last :t1) (last :t2) :w1 :w2][show [act8] act8 :a1 :b1
:d (last :t1) (last :t2) :w1 :w2]]]]]]]
end

to greq :a :b
op or :a > :b :a = :b
end

to leq :a :b
op or :a < :b :a = :b
end

to cond1 :t1 :t2
op (and ( leq 90 last :t1)  (leq 90 last :t2 )  (not (first
:t1) = (first :t2)))
end

to cond2 :t1 :t2
op (and ( leq 90 last :t1)  (90 < last :t2 )  ( (first :t1) =
(first :t2)))
end

to cond3 :t1 :t2
op (and ( greq 90 last :t1 )  (leq 90 last :t2)  ( (first :t1)
= (first :t2)))
end
```

```
to cond4 :t1 :t2
op (and (90 > last :t1)  (90 < last :t2)  ( not (first :t1) =
(first :t2)))
end

to cond5 :t1 :t2
op (and (greq 90 last :t1)  (greq 90 last :t2)  ( not (first
:t1) = (first :t2)))
end

to cond6 :t1 :t2
op (and (greq 90 last :t1)  (90 > last :t2)  (  (first :t1) =
(first :t2)))
end

to cond7 :t1 :t2
op (and (90 < last :t1)  (90 > last :t2)  ((first :t1) =
(first :t2)))
end

to cond8 :t1 :t2
op (and (90 < last :t1)  (90 > last :t2)  ( not (first :t1) =
(first :t2)))
end

to act1 :a1 :b1 :d  :t1 :t2 :w1 :w2
repeat 20 [ask :a1 [fd (-1 * :w1 * :d * sin :t2) / (20 * sin
(-180 - :t1  - :t2)) ] ask :b1 [fd  (-1 * :w2 * :d * sin :t1)
/ (20 * sin (-180 - :t1  - :t2))]]
end

to act2 :a1 :b1 :d  :t1 :t2 :w1 :w2
local "jw
make "jw bothway :a1 :b1
ifelse :jw = -1 [repeat 20 [ask :a1 [fd (-1  * :d * sin :t2) /
(20 * sin ( 180 - :t1 - 180 + :t2)) ] ask :b1 [fd  ( :d * sin
:t1) / (20 * sin ( 180 - :t1 - 180 + :t2)) ]]] [repeat 20 [ask
:a1 [ fd ( -1 * :d * sin :t2) / (20 * sin (:t1 - :t2)) ] ask
:b1 [fd  ( :d * sin :t1) / (20 * sin ( :t1 - :t2)) ]]]
end

to act3 :a1 :b1 :d  :t1 :t2 :w1 :w2
repeat 20 [ask :a1 [fd (:w1 * :d * sin :t2) / ( 20 * sin (:t1
+ 180 - :t2))] ask :b1 [fd  (:w2 * :d * sin :t1) / (20 * sin
(:t1 + 180 - :t2))]]
end

to act4 :a1 :b1 :d  :t1 :t2 :w1 :w2
repeat 20 [ask :a1 [fd (-1 * :w1 * :d * sin :t2) / ( 20 * sin
(:t1 - 180 + :t2))] ask :b1 [fd (:w2 * :d * sin :t1) / (20 *
sin (:t1 - 180 + :t2))]]
end
```

```
to act5 :a1 :b1 :d  :t1 :t2 :w1 :w2
repeat 20 [ask :a1 [fd (:w1 * :d * sin :t2) / (20 * sin (:t1 +
:t2))] ask :b1 [fd  (:w2 * :d * sin :t1) / ( 20 * sin (:t1 +
:t2))]]
end

to act6 :a1 :b1 :d  :t1 :t2 :w1 :w2
local "jw
make "jw bothway :a1 :b1
ifelse :jw = -1 [repeat 20 [ask :a1 [fd (:d * sin :t2) / (20 *
sin ( :t2 - :t1)) ] ask :b1 [fd  (-1 *  :d * sin :t1) / (20 *
sin (:t2 - :t1)) ]]] [repeat 20 [ask :a1 [fd (-1 * :d * sin
:t2) / (20 * sin ( :t1 - :t2)) ] ask :b1 [fd  (:d * sin :t1) /
(20 * sin (:t1 - :t2)) ]]]
end

to act7 :a1 :b1 :d  :t1 :t2 :w1 :w2
local "jw
make "jw bothway :a1 :b1
ifelse :jw = -1 [repeat 20 [ask :a1 [fd (-1 * :w1 * :d * sin
:t2) / (20 * sin ( 180 - :t1 - 180 + :t2)) ] ask :b1 [fd  (:w2
* :d * sin :t1) / (20 * sin ( 180 - :t1 - 180 + :t2)) ]]]
[repeat 20 [ask :a1 [fd (:w1 * :d * sin :t2) / (20 * sin ( :t1
- :t2)) ] ask :b1 [fd  (:w2 * :d * sin :t1) / (20 * sin (:t1 -
:t2))]]]
end

to act8 :a1 :b1 :d  :t1 :t2 :w1 :w2
repeat 20 [ask :a1 [fd (-1 * :w1 * :d * sin :t2) / (20 * sin
(:t1 + :t2))] ask :b1 [fd  (:w2 * :d * sin :t1) / ( 20 * sin
(:t1 + :t2))]]
end
```

;;;move first foward and second back

```
to bothway :a :b
local "f
ask  :a [ht pu fd 10]
ask :b  [ht pu bk 10]
make "f ask :a [distance :b]
ask :a [ht bk 20]
ask :b [ht fd 20]
ifelse (ask :a [distance :b]) > :f [make "f -1][make "f 1]
ask :b [bk 10  st]
ask :a [fd 10  st]
op :f
end

to sameheading? :a :b
op (abs(ask :a [heading]) - (ask :b [heading])) < 1.1
end
```

;;;remembering measures

```
to remember :x
make "m :m + 1
make (word "m :m) :x
memorybank, print (list word ":m :m :x)
end

to forget
memorybank, ct
end
```

;;; find the distance of a turtle from the mirror line

```
to shortestdist
local "w
make "w who
op abs round((sin (ask "mirror [towards :w])) * (ask "mirror
[distance :w]))
end
```

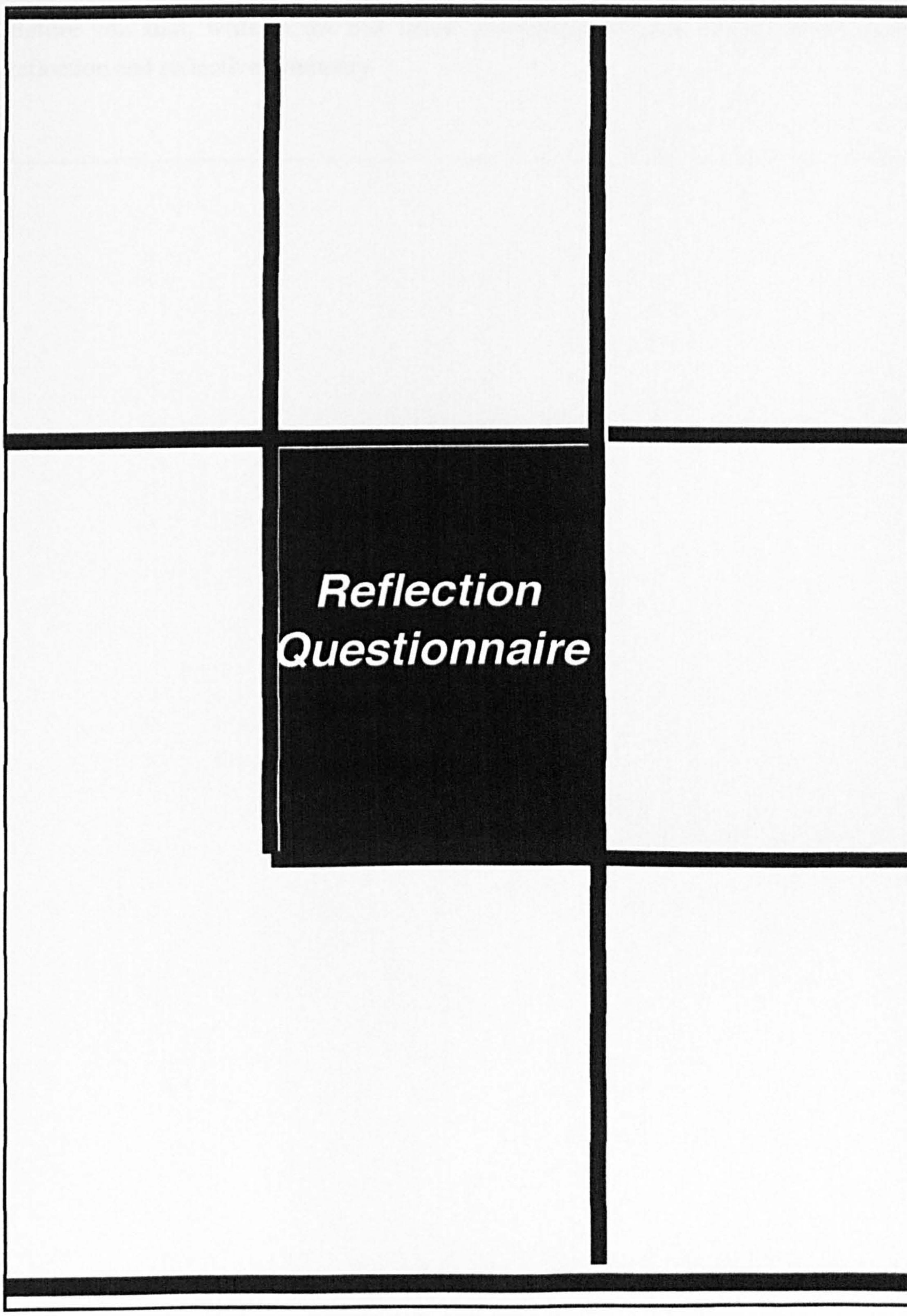
;;; other tools defined in -TOOLS- to change towards and distance and define face

```
to newfd :x
oldfd :x logit se "fd :x
end

to newbk :x
oldbk :x logit se "bk :x
end

to newrt :x
oldrt :x * (ask who [orientation])
logit se "rt :x
end

to newlt :x
oldlt :x * (ask who [orientation])
logit se "lt :x
end

to newseth :x
oldseth :x logit se "seth :x
end

to face :x
logit turnsize :x oldtowards :x
end

to newpu
oldpu logit  "pu
end
```

```
to newpd
oldpd logit "pd
end

to turnsize :x
local [a b]
make "memory :memory + 1
make "a heading
ht oldtowards :x make "b heading - :a
seth :a
make "memory :memory - 1 st
ifelse :b > 0 [op twiddlea :b ][op twiddleb :b]
end

to twiddlea :x
ifelse :x > 180 [op se "lt 360 - :x] [op se "rt :x]
end

to twiddleb :x
ifelse :x < -180 [op se "rt  360 - abs :x] [op se "lt abs :x]
end

to newtowards :x
ifelse :x = who [stop][ifelse (ask :x [pos]) = (ask who [pos])
[op toheading :x] [op turnsize :x]]
end

to newdistance :x
op round olddistance :x
end

to logit :list
llogit :list
end

to .startup
.ask "sys [copydef "core "fd "core "oldfd]
.ask "sys [copydef "core "newfd "core "fd]
.ask "sys [copydef "core "bk "core "oldbk]
.ask "sys [copydef "core "newbk "core "bk]
.ask "sys [copydef "core "rt "core "oldrt]
.ask "sys [copydef "core "newrt "core "rt]
.ask "sys [copydef "core "lt "core "oldlt]
.ask "sys [copydef "core "newlt "core "lt]
.ask "sys [copydef "core "pu "core "oldpu]
.ask "sys [copydef "core "newpu "core "pu]
.ask "sys [copydef "core "pd "core "oldpd]
.ask "sys [copydef "core "newpd "core "pd]
.ask "sys [copydef "core "seth "core "oldseth]
.ask "sys [copydef "core "newseth "core "seth]
.ask "sys [copydef "core "towards "core "oldtowards]
.ask "sys [copydef "core "newtowards "core "towards]
.ask "sys [copydef "core "distance "core "olddistance]
.ask "sys [copydef "core "newdistance "core "distance]
end
```

Appendix 3

Final version of the paper and pencil test

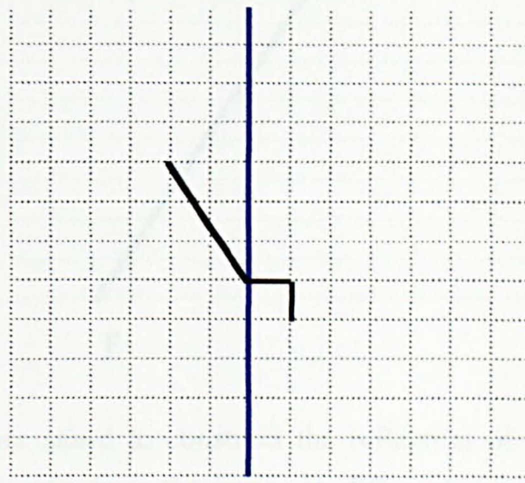


You are going to complete a survey that is all about reflective symmetry.

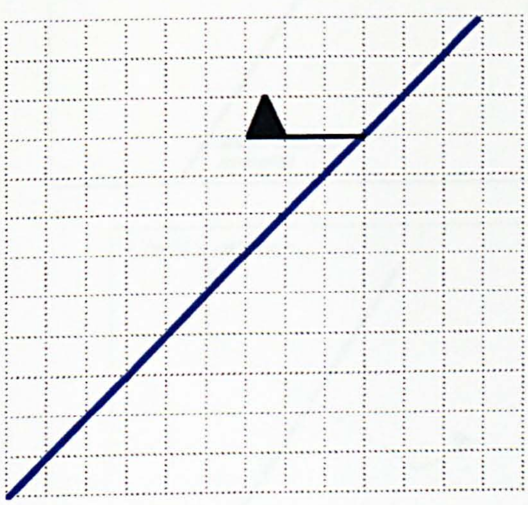
Before you start, write in the box below everything that you already know about reflection and reflective symmetry

1. Using the thick line as the line of symmetry (the mirror line) carefully sketch the reflections of the following shapes:

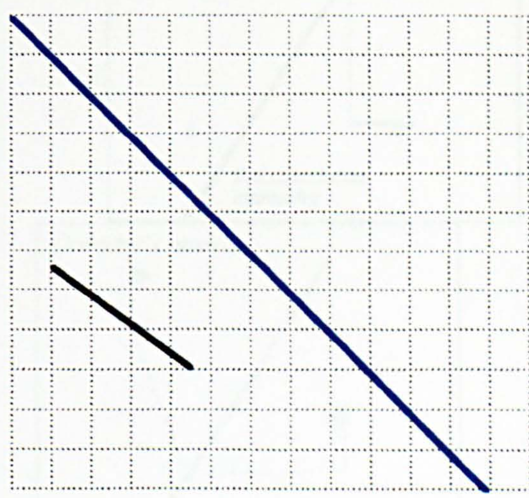
A



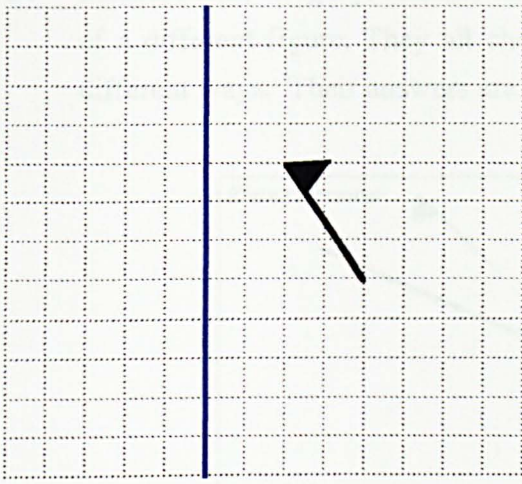
B



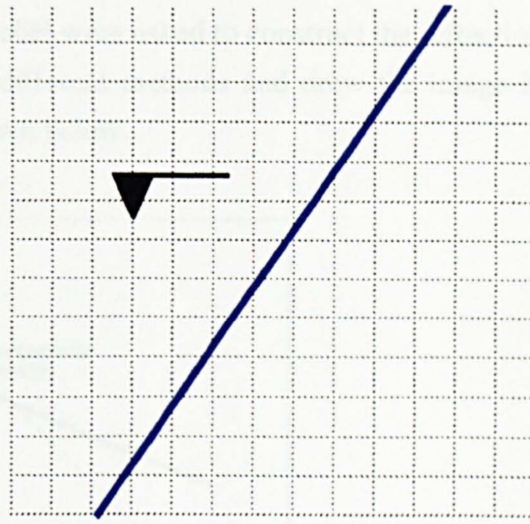
C



D

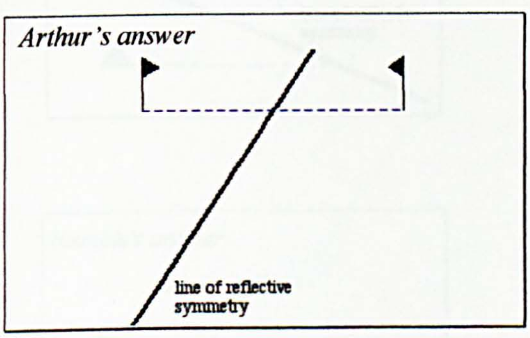


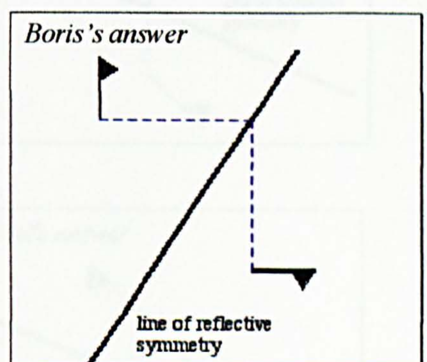
E

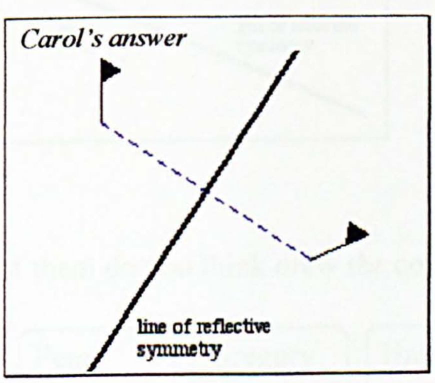


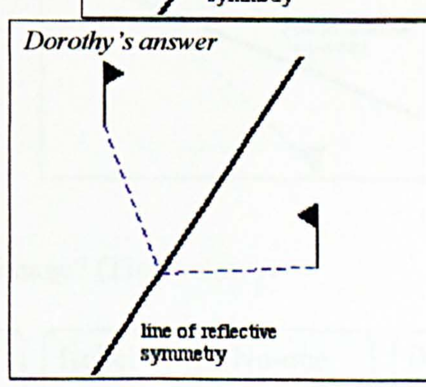
F

2. Anthony, Boris, Carol and Dorothy were asked to construct the reflection of a figure. They all chose different methods and drew the image in different ways. Their answers are shown below.





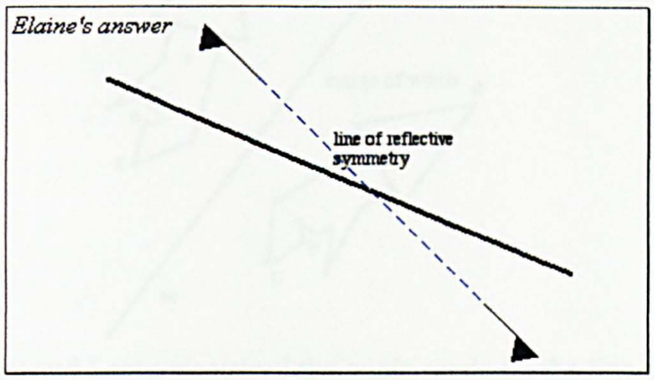


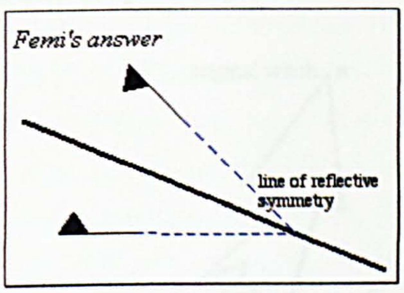


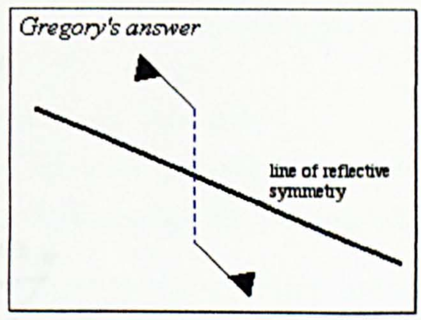
Which of them do you think drew the correct image? (Tick below)

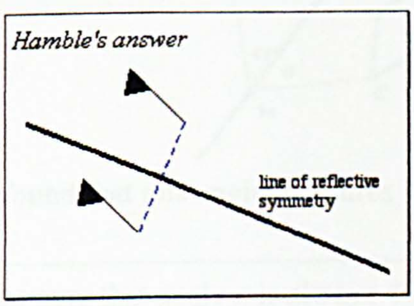
Anthony	Boris	Carol	Dorothy	No-one	Don't know

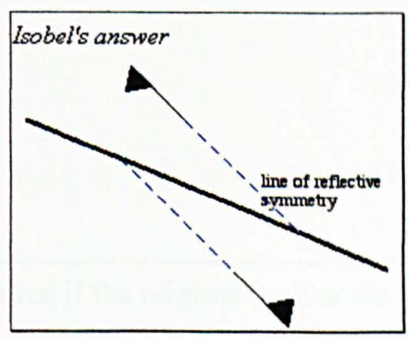
3. Elaine, Femi, Gregory, Hamble and Isobel were asked to construct the reflection of a different figure. They all chose different methods and drew the image in different ways. Their answers are shown below.







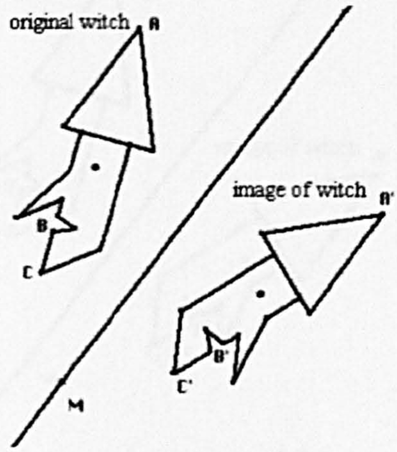




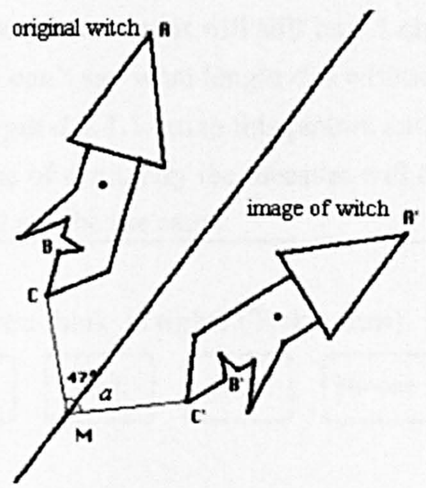
Which of them do you think drew the correct image? (Tick below)

Elaine	Femi	Gregory	Hamble	Isobel	No-one	Don't know

4. The figure below shows the reflection of a witch's head.



- Jemina, Kerry and Larry measured the angle made by the line segment CM and the line of symmetry.



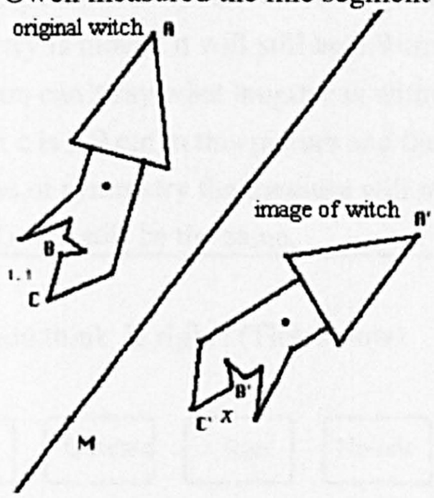
They found that this angle measures 47°.

Jemima says that angle *a* is always 47° and that even if the original head or the line of symmetry is moved it will still be 47°. *Kerry* says that you can't say what angle *a* is without measuring it. *Larry* says that angle *a* is 47° in this picture and that if you move the original head or the line of symmetry the measure will change but the two angles will still be the same.

Which of them do you think is right? (Tick below)

Jemima	Kerry	Larry	No-one	Don't know

❑ Nick, Mick and Owen measured the line segment CB.



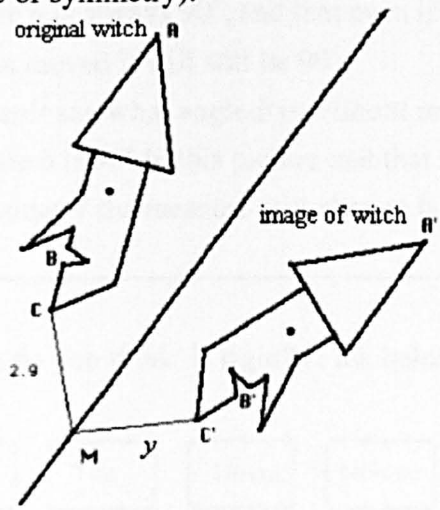
They found that this line segment measures 1.1 cm

Nick says that length *d* is always 1.1 cm, and that even if the original head or the line of symmetry is moved it will still be 1.1 cm. *Mick* says that you can't say what length *d* is without measuring it. *Owen* says that length *d* is 1.1 cm in this picture and that if you move the original head or the line of symmetry the measure will change but the two lengths (CB and C'B') will still be the same.

Which of them do you think is right? (Tick below)

Nick	Mick	Owen	No-one	Don't know

❑ Pauline, Queenie and Rob measured the distance CM (from the witch's chin to a point on the line of symmetry).



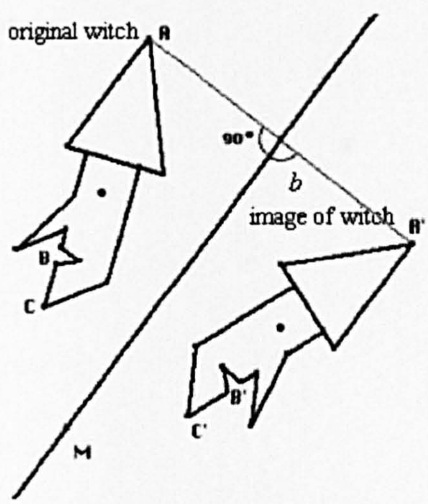
They found that the distance is 2.9 cm.

Pauline says that length *c* is always 2.9 cm, and that even if the original head or the line of symmetry is moved it will still be 2.9 cm.
Queenie says that you can't say what length c is without measuring it.
Rob says that length c is 2.9 cm in this picture and that if you move the original head or the line of symmetry the measure will change but the two lengths (CM and C'M) will still be the same.

Which of them do you think is right? (Tick below)

Pauline	Queenie	Rob	No-one	Don't know

- Stella, Tim and Ulrika drew the line segment AA'.



They found that this line segment crosses the line at symmetry at 90°.

Stella says that angle *b* is always 90°, and that even if the original head or the line of symmetry is moved it will still be 90°.
Tim says that you can't say what angle *b* is without measuring it.
Ulrika says that angle *b* is 90° in this picture and that if you move original head or the line of symmetry the measure will change but the two angles will still be the same.

(iv) Which of them do you think is right? (Tick below)

Stella	Tim	Ulrika	No-one	Don't know

5. In the following diagram there are two points. One point is the reflection of the other, but the line of symmetry that was used in the reflection is missing. Reconstruct the line as symmetry as accurately as you can.

In the box below, describe how you constructed your line of symmetry

Appendix 4

Students' initial profiles

The following descriptions indicate how these types were drawn up in terms of students' profile of responses the paper-and-pencil test[1] and the number of students within each type that were chosen:

Type I, typical: Two students with a profile of typical responses were chosen for each group. Typical was defined in the following way: in items where over 50% of the responses fell into one category, typical students gave this response. On questions where they most common response was given by less than 50%, the responses of typical students corresponded to one of the top two responses.

Type II, perceptual: One student was chosen whose responses suggested a good visual feel for reflection in two dimensions. This student type was characterised by responses involving visual products that were more or less correct, but that did not present explicit clues of any method of construction. Students falling into this type also tended to give confused responses (or no responses at all) to the fourth part of the test.

Type III, theoretical: A fourth student was chosen whose script suggested constructions and choices had been made on a theoretical basis. They were particularly distinguishable from other students types by the use of orientation/angle properties in systematic ways in their constructions, although the images they then produced were not always indicative a good visual feel for 2-D reflections. Because of the small number of students who did this, this type included those who explicitly referred to either the perpendicular or equal angle properties of reflection. Properties of their construction were made explicit either through marked construction lines or in

[1] The written descriptions of reflection and reflective symmetry from the paper and pencil tests were not used in the determination of particular student types and did not seem to vary in particular ways according to them.

an accompanying description. A further condition of selection into this group was that at least one of the angle related questions in the fourth part of the test was correctly answered.

Type IV, extreme interfigural: The fifth and sixth members of the group were selected so that the group would include two students who accepted images produced by other isometric transformations as the images of reflection in two dimensions. These students were chosen on the basis of their responses to the second and third parts of the tests.

The initial profiles of the twenty-four students selected to participated in the four learning systems are displayed in Tables A4.1-A4.24.

Profiles of the DEG-FI students

<table>
<tr><td rowspan="2">Rhea
(Type I)</td><td>Description

Congruency
Reversed
Division of space

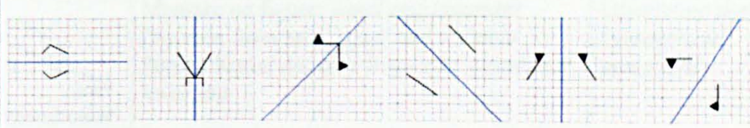
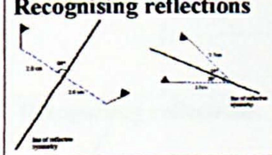
Perceived object/ Property</td><td colspan="2">Drawing reflections
</td></tr>
<tr><td>Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: equal and invariant
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.1: Profile of Rhea (Type I student DEG-FI)

<table>
<tr><td rowspan="2">Anita
(Type I)</td><td>Description

Congruency
Mirror

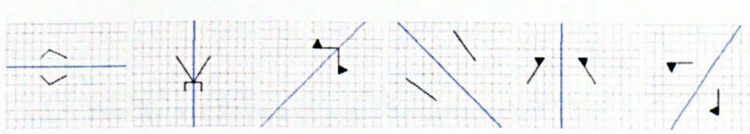
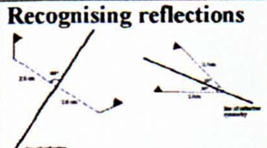
Property</td><td colspan="2">Drawing reflections
</td></tr>
<tr><td>Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: equal and invariant
Distance from axis: equal but variable
Perpendicular distance from axis: unknown</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.2: Profile of Anita (Type I student DEG-FI)

<table>
<tr><td rowspan="2">Suzie
(Type II)</td><td>Description

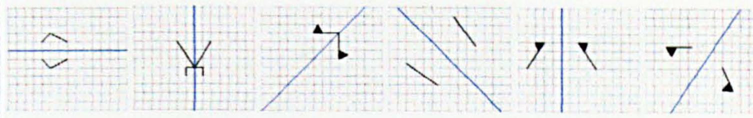
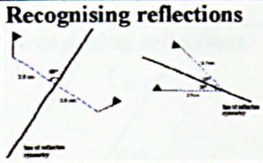
Congruency
Perceived object</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: equal but variable
Distance from axis: equal and invariant
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.3: Profile of Suzie (Type II student DEG-FI)

<table>
<tr><td rowspan="2">Sharmila
(Type III)</td><td>Description

Congruency
Mirror
Division of space

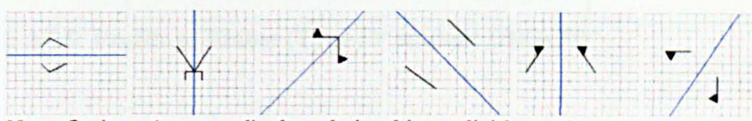
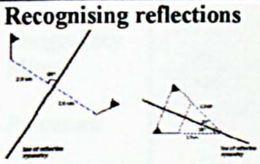
Physical process</td><td colspan="3">Drawing reflections

Note: On item 4, perpendicular relationship explicitly used</td></tr>
<tr><td colspan="2">Recognising reflections

Note: added perpendicular lines</td><td>Identifying properties
Angle measure: no response
Measure on figure: equal and invariant
Distance from axis: equal and variable
Perpendicular distance from axis: equal and invariant</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.4: Profile of Sharmila (Type III student DEG-FI)

<table>
<tr><td rowspan="2">Christie
(Type III)</td><td>Description

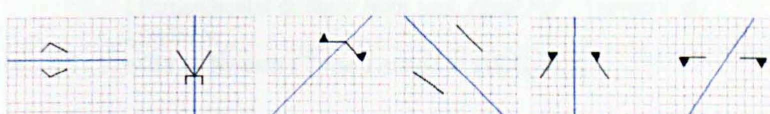
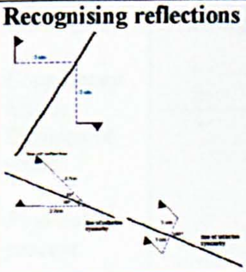
Congruency</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: equal but variable
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Point joined</td></tr>
</table>

Table A4.5: Profile of Christie (Type IV student DEG-FI)

<table>
<tr><td rowspan="2">Elaine
(Type III)</td><td>Description

Congruency
Mirror
Division of space

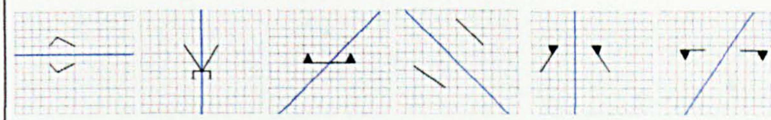
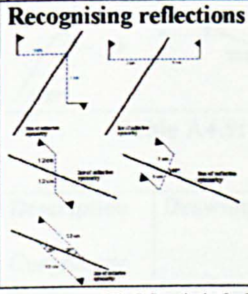
Physical process</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: equal and invariant
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.6: Profile of Elaine (Type IV student DEG-FI)

Profiles of the DEG-FO students

<table>
<tr><td rowspan="2">Sita
(Type I)</td><td>Description
Congruency
Mirror

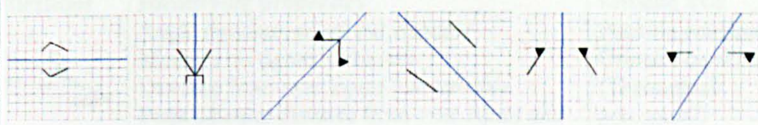
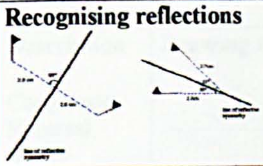
Perceived object</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: equal and invariant
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.7: Profile of Seema (Type I student DEG-FO)

<table>
<tr><td rowspan="2">Rebekka
(Type I)</td><td>Description
Congruency
Mirror
Division of space

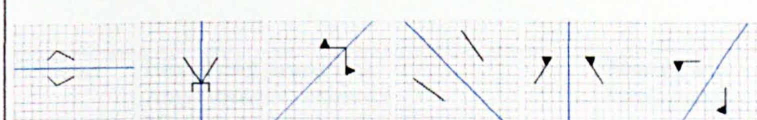
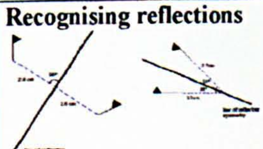
Physical process property</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: equal but variable
Distance from axis: equal but variable
Perpendicular distance from axis: unknown</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.8: Profile of Rebekka (Type I student DEG-FO)

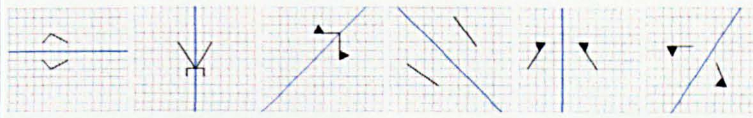
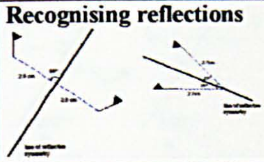
Anju (Type II)	**Description** Congruency *Property*	**Drawing reflections** 	
	Recognising reflections 	**Identifying properties** Angle measure: *unknown* Measure on figure: *equal but variable* Distance from axis: *equal and invariant* Perpendicular distance from axis: *no response*	**Missing axis** Position sketched Orientation sketched (Perceptual)

Table A4.9: Profile of Anju (Type II student DEG-FO)

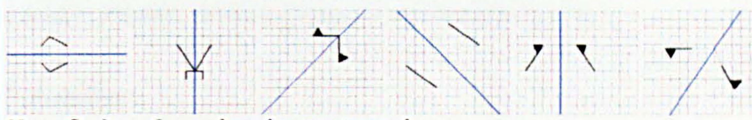
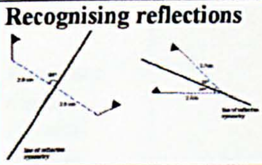
Seema (Type III)	**Description** Congruency Mirror Division of space *Perceived object*	**Drawing reflections**  *Note:* On item 6, equal angles constructed	
	Recognising reflections 	**Identifying properties** Angle measure: *equal but variable* Measure on figure: *unknown* Distance from axis: *equal but variable* Perpendicular distance from axis: *equal and invariant*	**Missing axis** Position constructed Orientation sketched (Theoretical/ perceptual)

Table A4.10: Profile of Sita (Type III student DEG-FO)

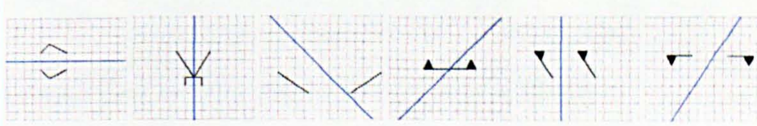
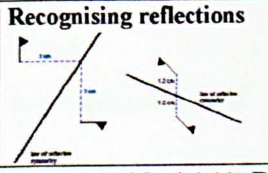
Kylie (Type III)	**Description** Congruency Reversal Mirror Division of sides *Physical process*	**Drawing reflections** 	
	Recognising reflections 	**Identifying properties** Angle measure: *unknown* Measure on figure: *unknown* Distance from axis: *unknown* Perpendicular distance from axis: *unknown*	**Missing axis** Points joined

Table A4.11: Profile of Kylie (Type IV student DEG-FO)

<table>
<tr><td rowspan="2">Maia
(Type III)</td><td>Description

Congruency
Mirror

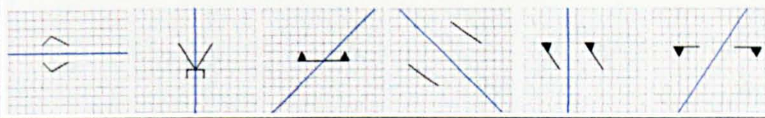
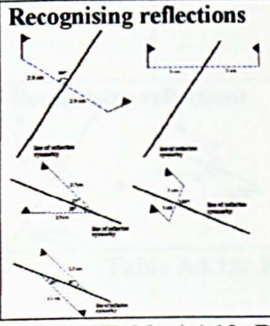
Property</td><td colspan="2">Drawing reflections
</td></tr>
<tr><td>Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: unknown
Distance from axis: unknown
Perpendicular distance from axis: unknown</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/
perceptual)</td></tr>
</table>

Table A4.12: Profile of Maia (Type IV student DEG-FO)

Profiles of the MTG-FI students

<table>
<tr><td rowspan="2">Hadley
(Type I)</td><td>Description

Congruency
Division of sides

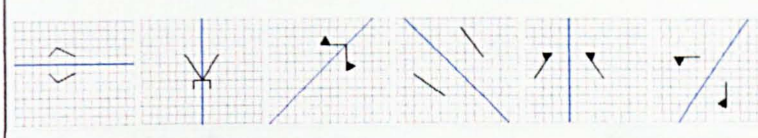
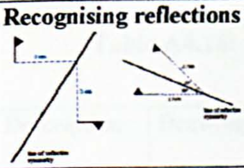
Perceived object
Physical process</td><td colspan="2">Drawing reflections
</td></tr>
<tr><td>Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: equal but variable
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/
perceptual)</td></tr>
</table>

Table A4.13: Profile of Hadley (Type I student MTG-FI)

<table>
<tr><td rowspan="2">Lizzie
(Type I)</td><td>Description

Congruency
Mirror

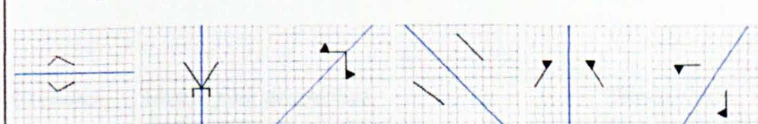
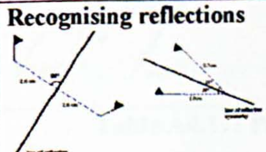
Property</td><td colspan="2">Drawing reflections
</td></tr>
<tr><td>Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: equal and invariant
Distance from axis: equal but variable
Perpendicular distance from axis: unknown</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/
perceptual)</td></tr>
</table>

Table A4.14: Profile of Lizzie (Type I student MTG-FI)

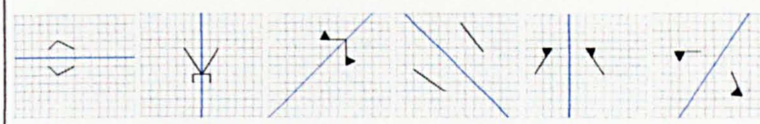
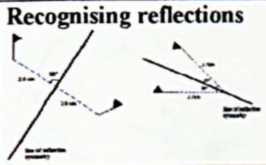
Alissa (Type II)	**Description** Congruency Mirror *Physical process*	**Drawing reflections** 	
	Recognising reflections 	**Identifying properties** Angle measure: *unknown* Measure on figure: *unknown* Distance from axis: *equal and invariant* Perpendicular distance from axis: *equal and variable*	**Missing axis** Position constructed Orientation sketched (Theoretical/ perceptual)

Table A4.15: Profile of Alissa (Type II student MTG-FI)

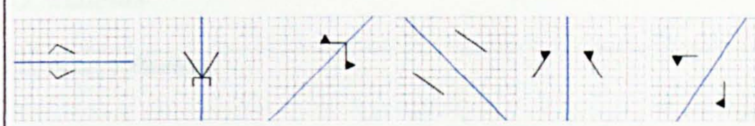
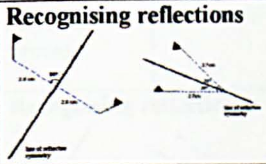
Lorna (Type III)	**Description** Congruency Mirror Reversal *Perceived object* *Property*	**Drawing reflections** 	
	Recognising reflections 	**Identifying properties** Angle measure: *equal but variable* Measure on figure: *equal and invariant* Distance from axis: *equal but variable* Perpendicular distance from axis: *equal but variable*	**Missing axis** Position constructed Orientation constructed (Theoretical)

Table A4.16: Profile of Lorna (Type III student MTG-FI)

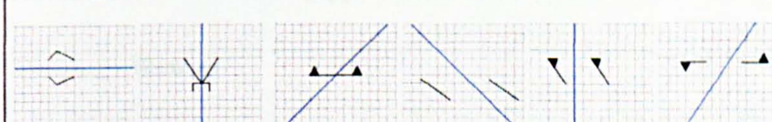
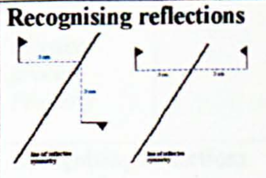
Aimee (Type IV)	**Description** Congruency Division of space *Property*	**Drawing reflections** 	
	Recognising reflections 	**Identifying properties** Angle measure: *no response* Measure on figure: *equal and variable* Distance from axis: *equal and variable* Perpendicular distance from axis: *equal and variable*	**Missing axis** Position sketched Orientation sketched (Perceptual)

Table A4.17: Profile of Aimee (Type IV student MTG-FI)

<table>
<tr><td rowspan="2">Helen
(Type IV)</td><td>Description

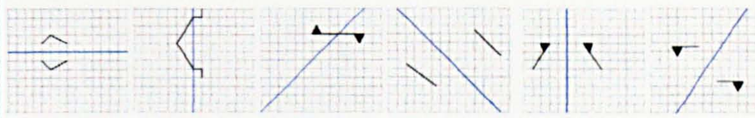
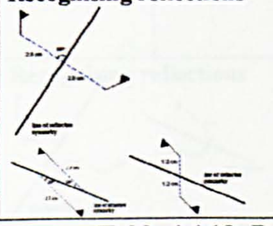
Mirror
Perceived object</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: equal but variable
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.18: Profile of Helen (Type IV student MTG-FI)

Profiles of the MTG-FO students

<table>
<tr><td rowspan="2">Laurel
(Type I)</td><td>Description

Congruency
Division of sides

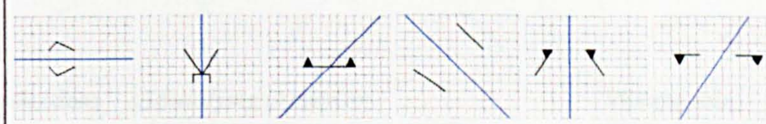
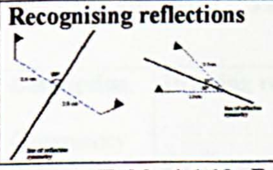
Perceived object
Physical process</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: equal and invariant
Distance from axis: equal but variable
Perpendicular distance from axis: unknown</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.19: Profile of Laurel (Type I student MTG-FO)

<table>
<tr><td rowspan="2">Candy
(Type I)</td><td>Description

Congruency
Mirror
Behaviour of light

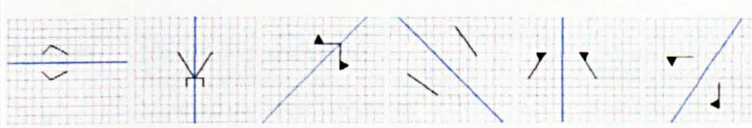
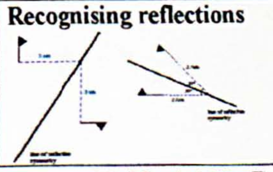
Physical process
Property</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: unknown
Measure on figure: equal but variable
Distance from axis: equal but variable
Perpendicular distance from axis: unknown</td><td>Missing axis
Position constructed
Orientation sketched
(Theoretical/ perceptual)</td></tr>
</table>

Table A4.20: Profile of Candy (Type I student MTG-FO)

<table>
<tr><td rowspan="2">Prija
(Type II)</td><td>Description

Congruency
Reversal

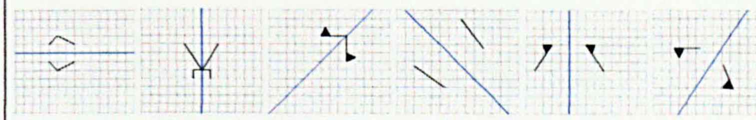
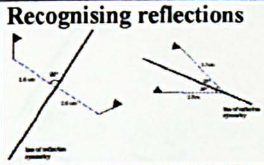
Physical process
Property</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: equal but variable
Distance from axis: equal and invariant
Perpendicular distance from axis: equal and variable</td><td>Missing axis
Position sketched
Orientation sketched
(Perceptual)</td></tr>
</table>

Table A4.21: Profile of Prija (Type II student MTG-FO)

<table>
<tr><td rowspan="2">Jodie
(Type III)</td><td>Description

Mirror

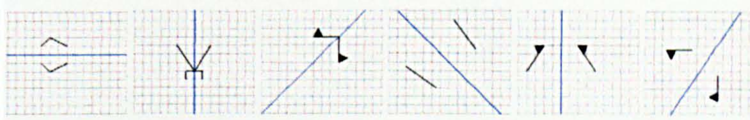
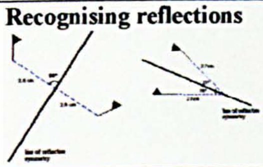
Perceived object</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: equal and invariant
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position constructed
Orientation constructed
(Theoretical)</td></tr>
</table>

Table A4.22: Profile of Jodie (Type III student MTG-FO)

<table>
<tr><td rowspan="2">Kerry
(Type IV)</td><td>Description

Congruency
Mirror

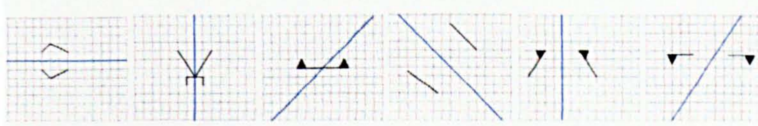
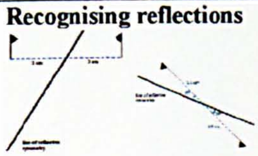
Physical process</td><td colspan="3">Drawing reflections
</td></tr>
<tr><td colspan="2">Recognising reflections
</td><td>Identifying properties
Angle measure: equal but variable
Measure on figure: unknown
Distance from axis: equal but variable
Perpendicular distance from axis: equal but variable</td><td>Missing axis
Position sketched
Orientation sketched
(Perceptual)</td></tr>
</table>

Table A4.23: Profile of Kerry (Type IV student MTG-FO)

<table>
<tr>
<td rowspan="2">Sophy
(Type IV)</td>
<td>Description

Congruency
Mirror

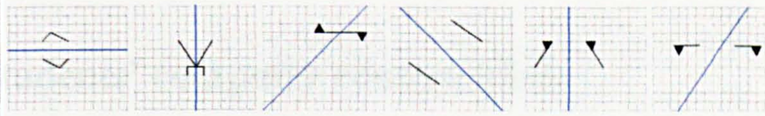
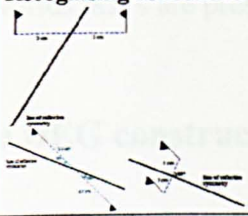
Physical process
Property</td>
<td colspan="3">Drawing reflections
</td>
</tr>
<tr>
<td>Recognising reflections
</td>
<td colspan="2">Identifying properties
Angle measure: no response
Measure on figure: equal and invariant
Distance from axis: equal and invariant
Perpendicular distance from axis: unknown</td>
<td>Missing axis
Points joined</td>
</tr>
</table>

Table A4.24: Profile of Sophy (Type IV student MTG-FO)

Appendix 5

Students' computer constructions

Students' computer constructions (as saved in a computer file) at the end of each of the microworlds tasks are presented in this appendix.

A5.1 The DEG constructions

This section contains figures of the final screen and the exposition of the students DEG interactions on all five DEG tasks.

A5.1.1 Final versions of computer constructions of Anita and Sharmila (type I-III pairing, DEG-FI)

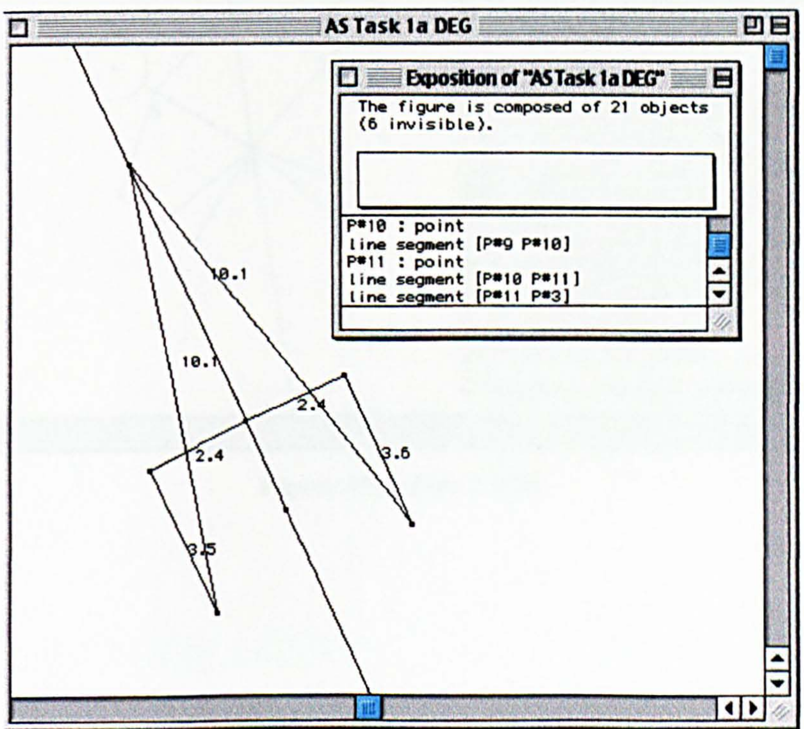


Figure A5.1:Task 1a DEG

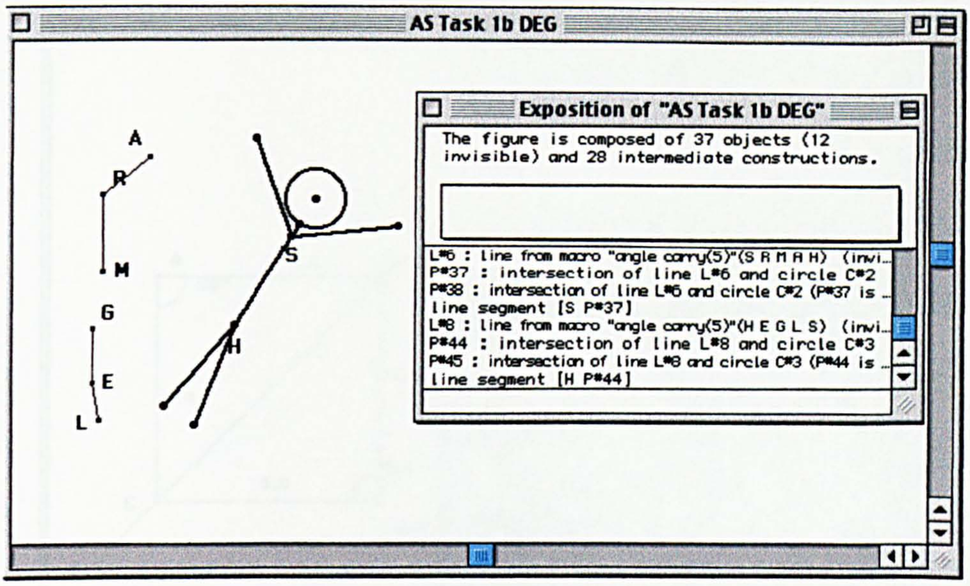


Figure A5.2:Task 1b DEG

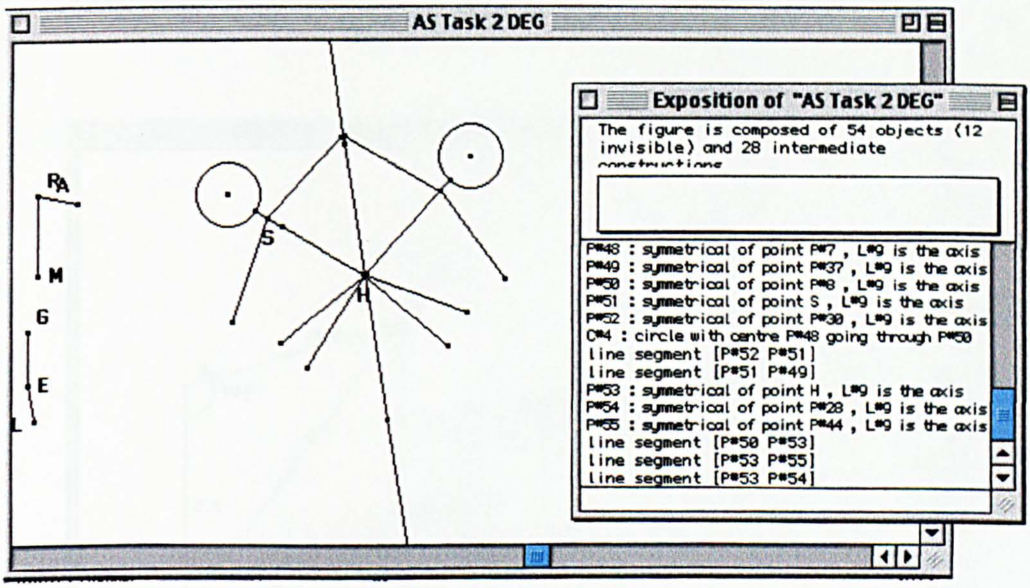


Figure A5.3:Task 2 DEG

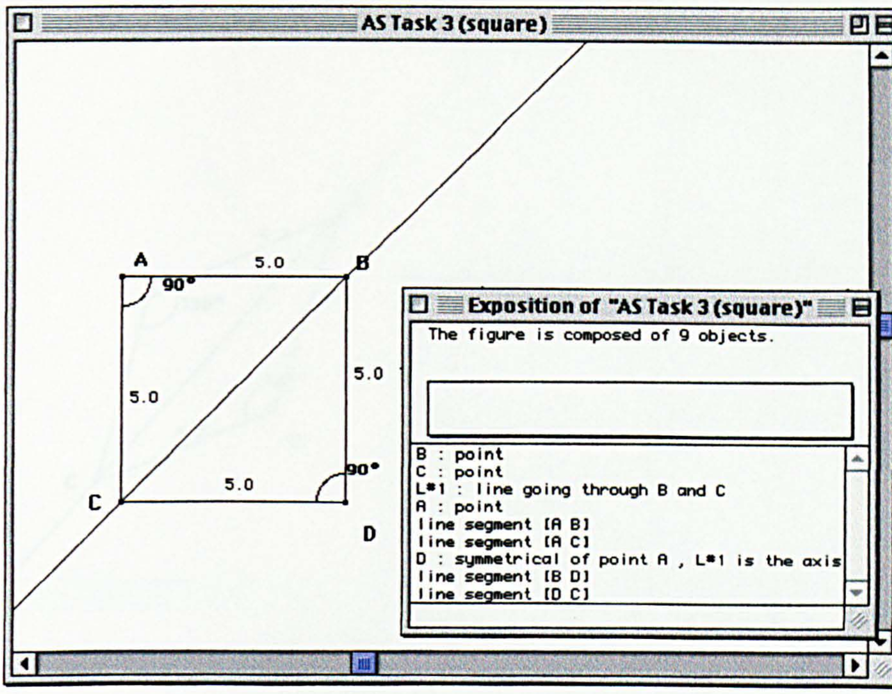


Figure A5.4:Task 3 (figure 1)

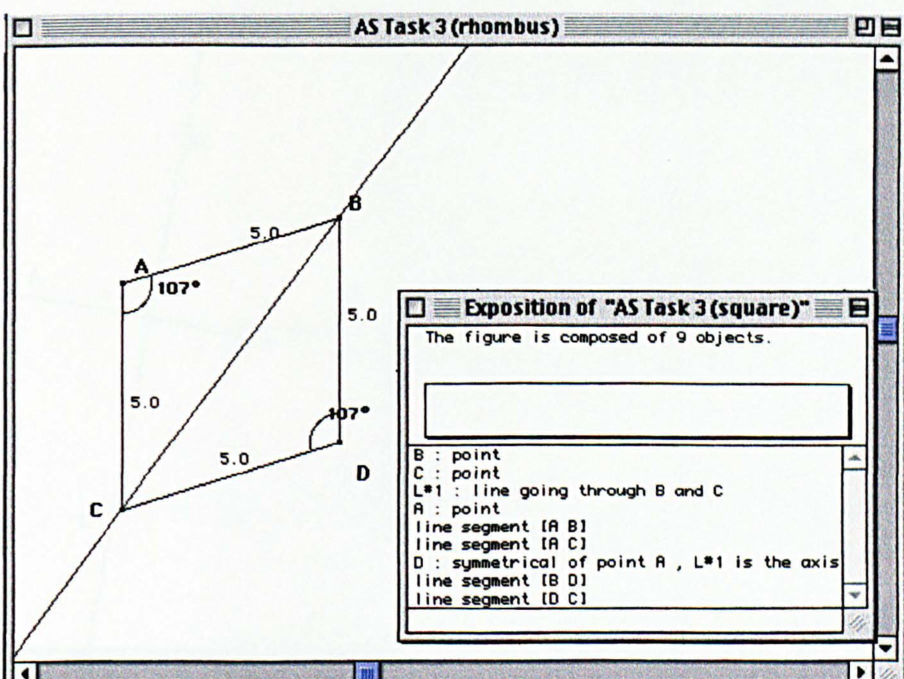


Figure A5.5:Task 3 DEG (figure 2)

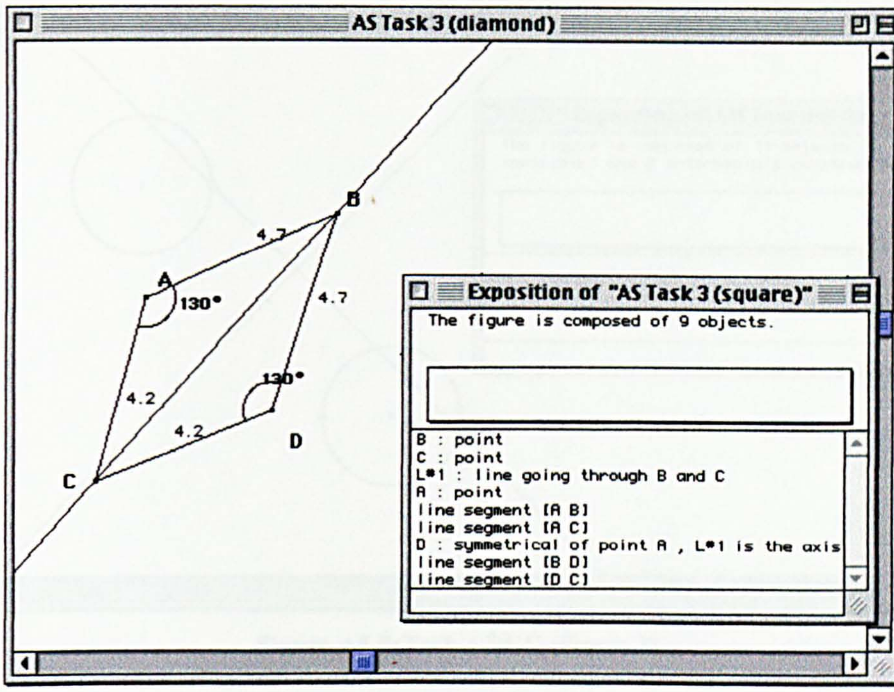


Figure A5.6:Task 3 DEG (figure 3)

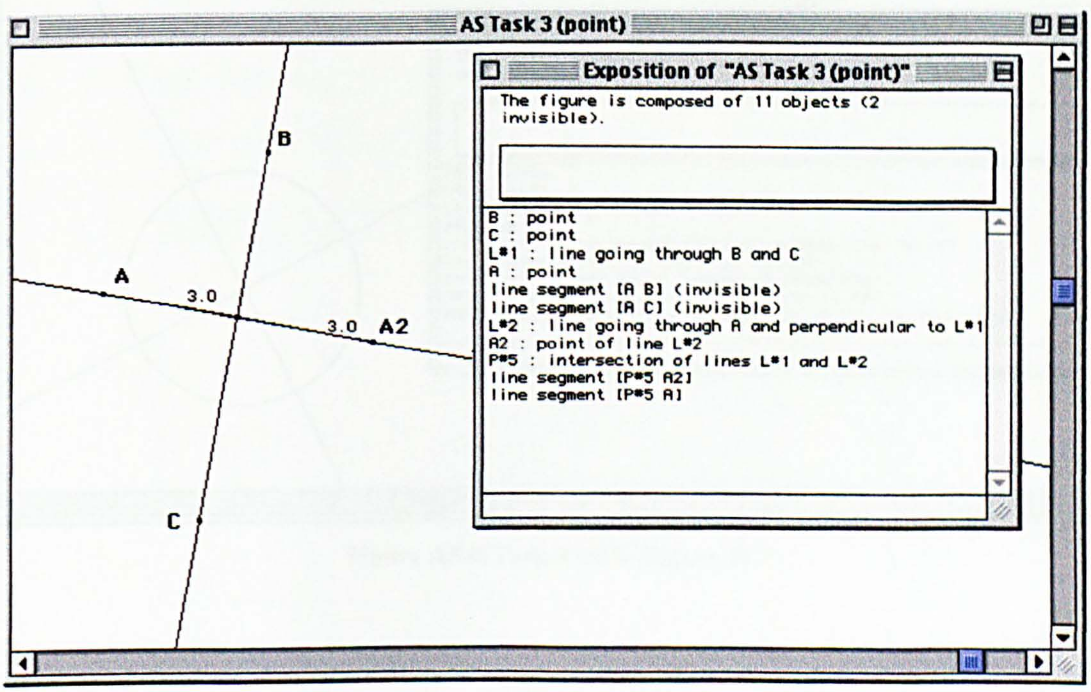


Figure A5.7:Task 3 DEG (figure 4)

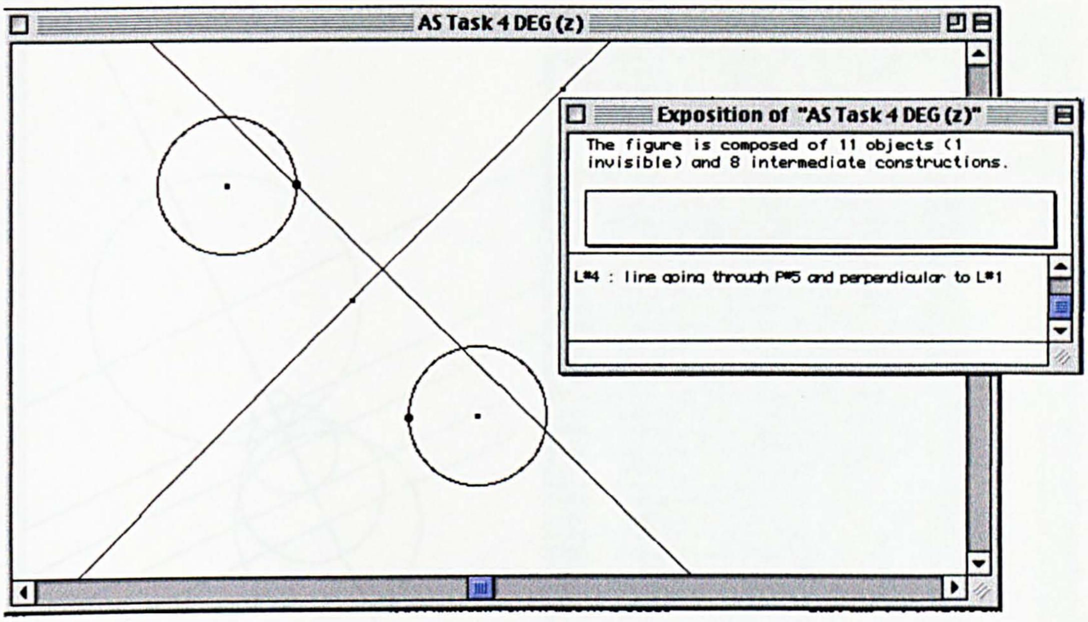


Figure A5.8:Task 4 DEG (figure 1)

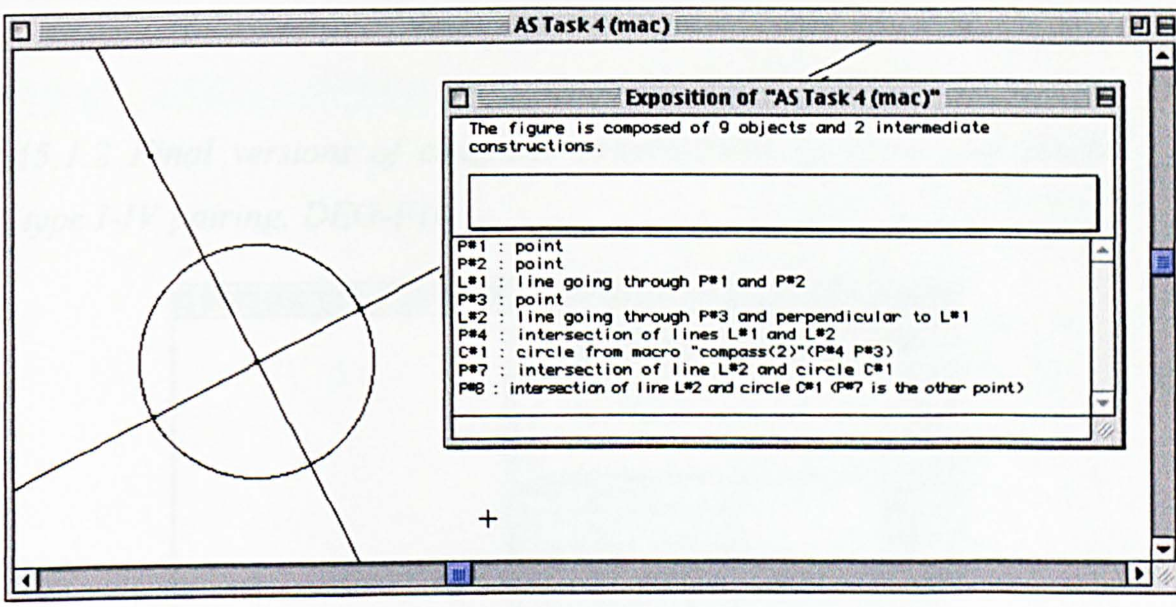


Figure A5.9:Task 4 DEG (figure 2)

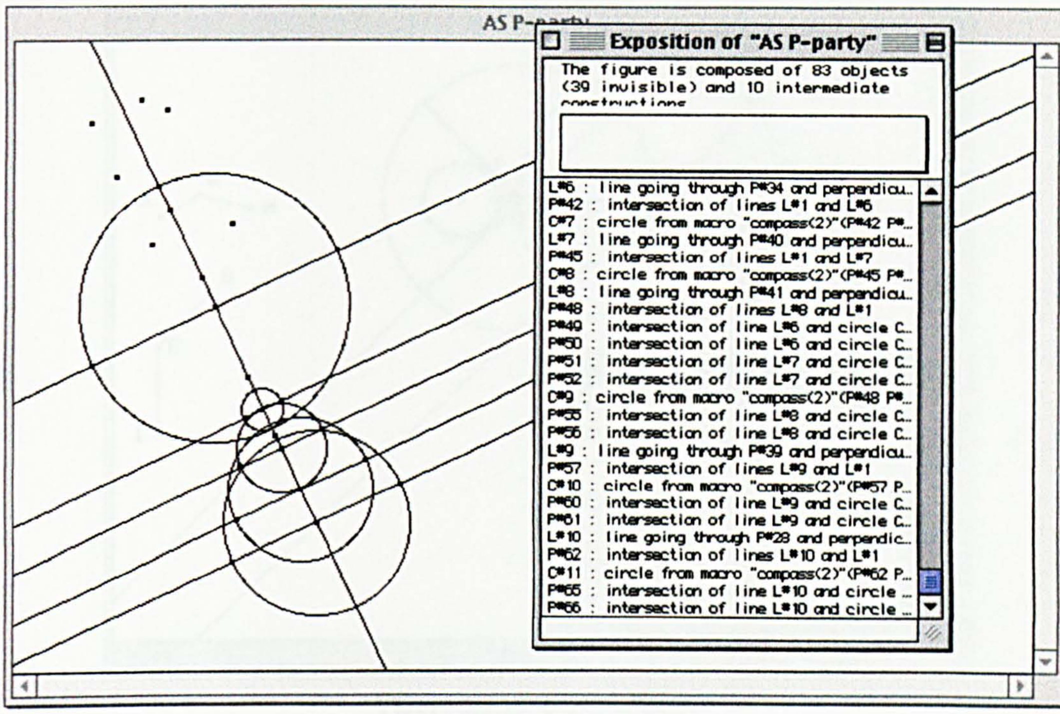


Figure A5.10:Task 5 DEG

A5.1.2 Final versions of computer constructions of Rhea and Elaine (type I-IV pairing, DEG-FI)

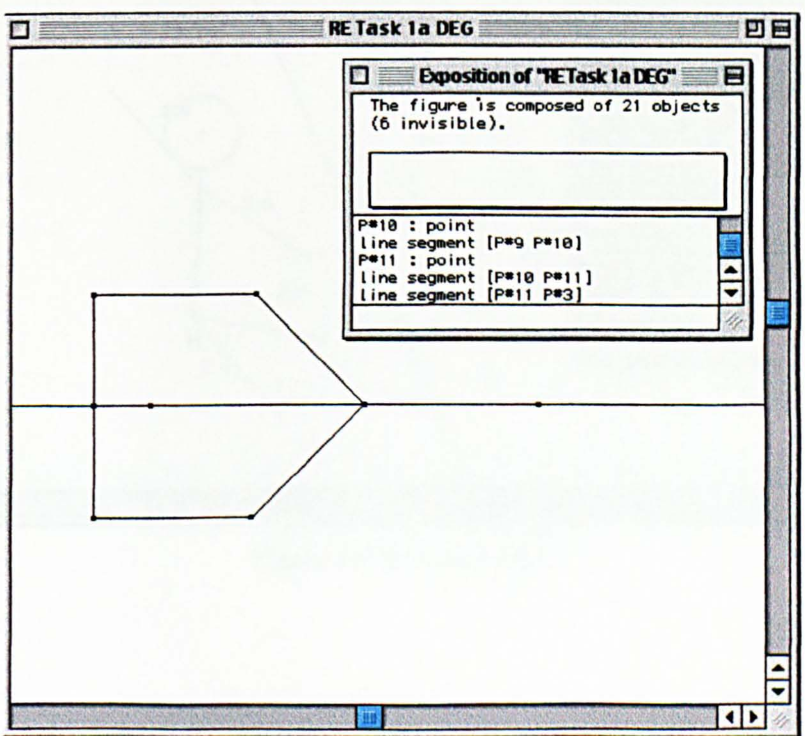


Figure A5.11:Task 1a DEG

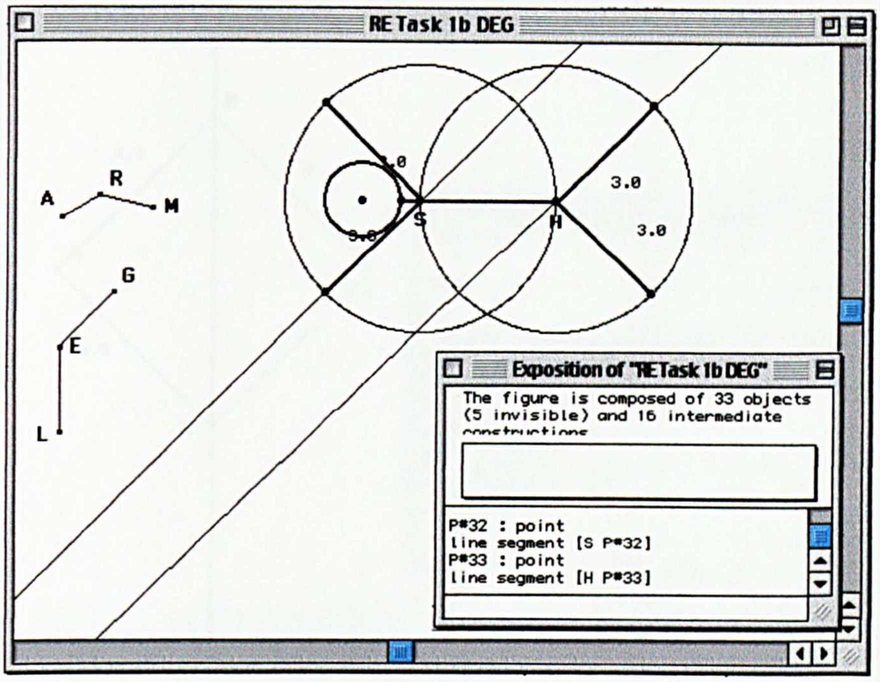


Figure A5.12:Task 1b DEG

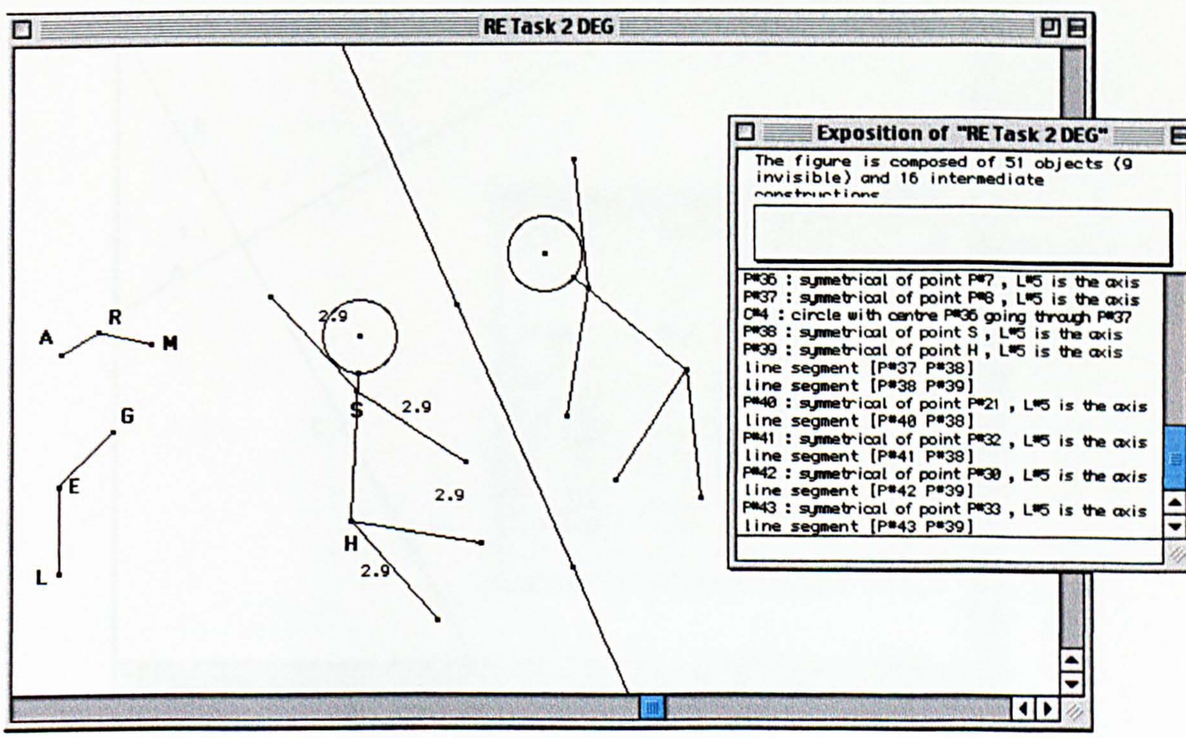


Figure A5.13:Task 2 DEG

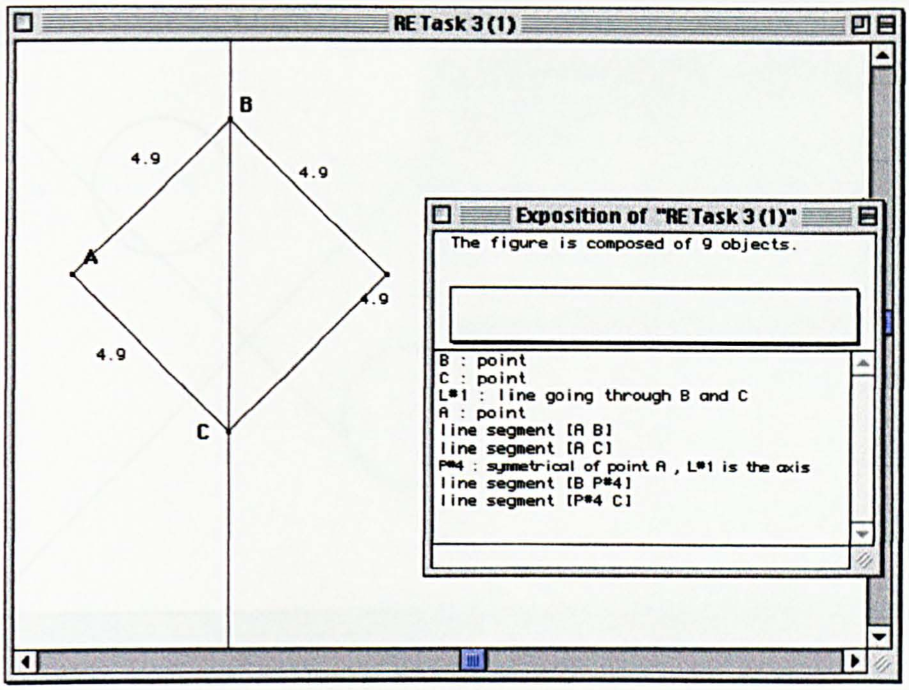


Figure A5.14:Task 3 (figure 1)

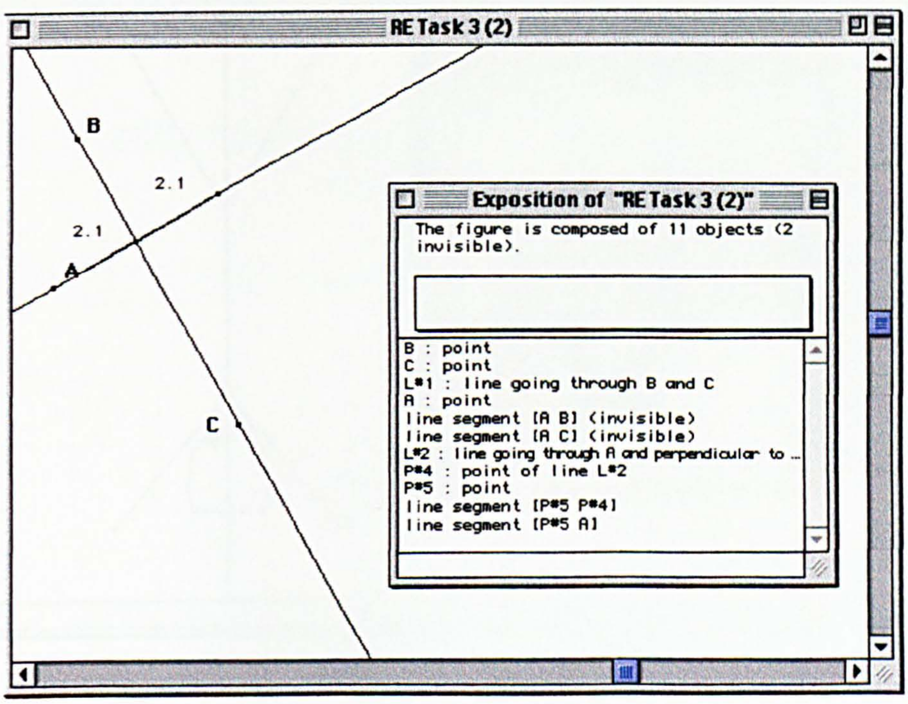


Figure A5.15:Task 3 DEG (figure 2)

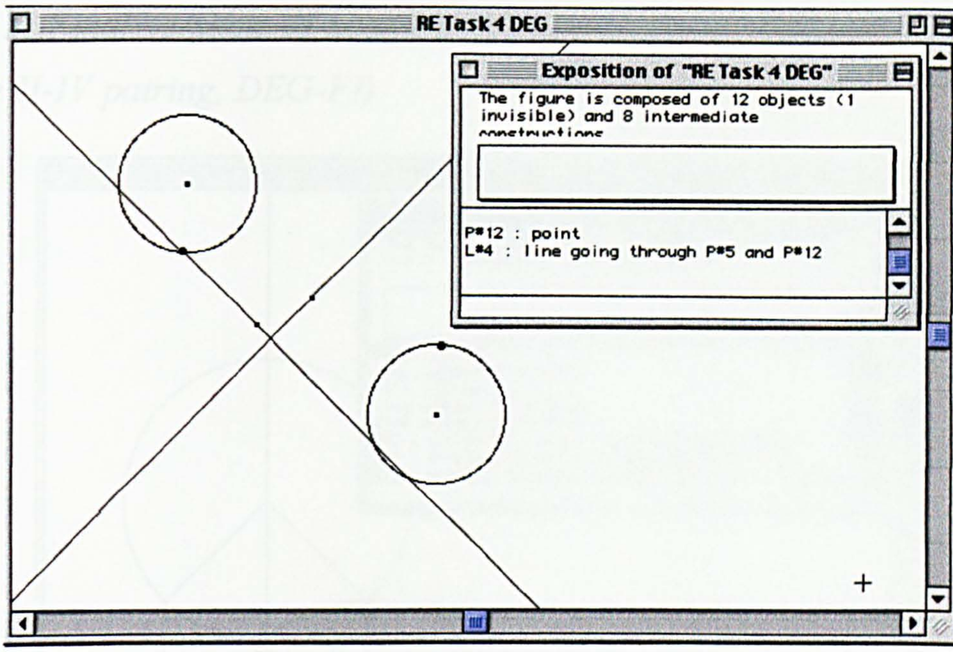


Figure A5.16:Task 4 DEG

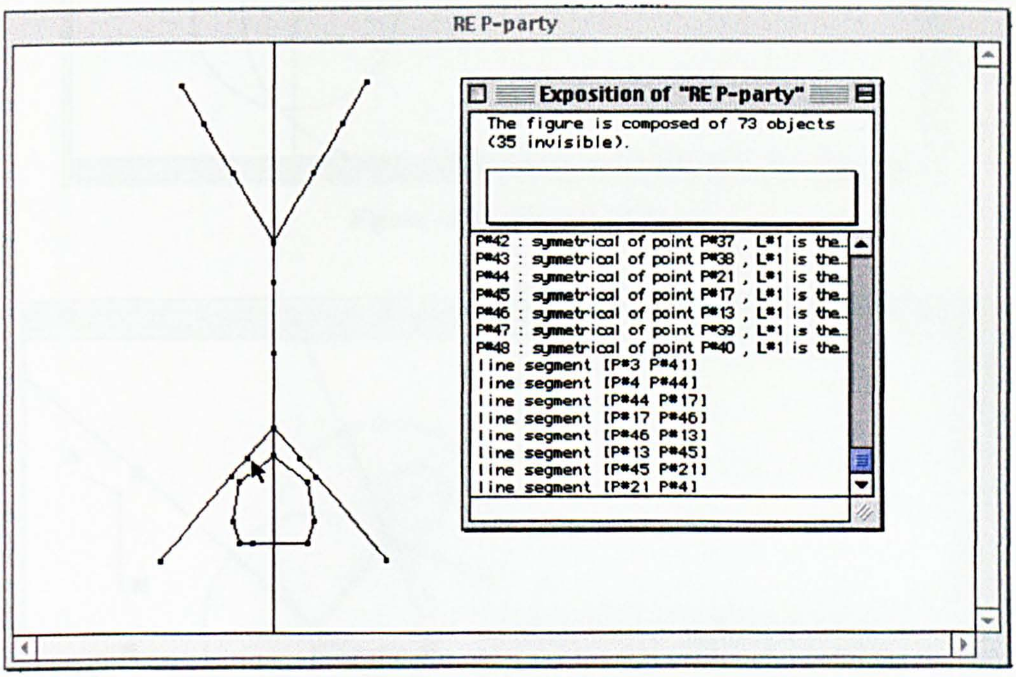


Figure A5.17:Task 5 DEG

A5.1.3 Final versions of computer constructions of Suzie and Christie (type II-IV pairing, DEG-FI)

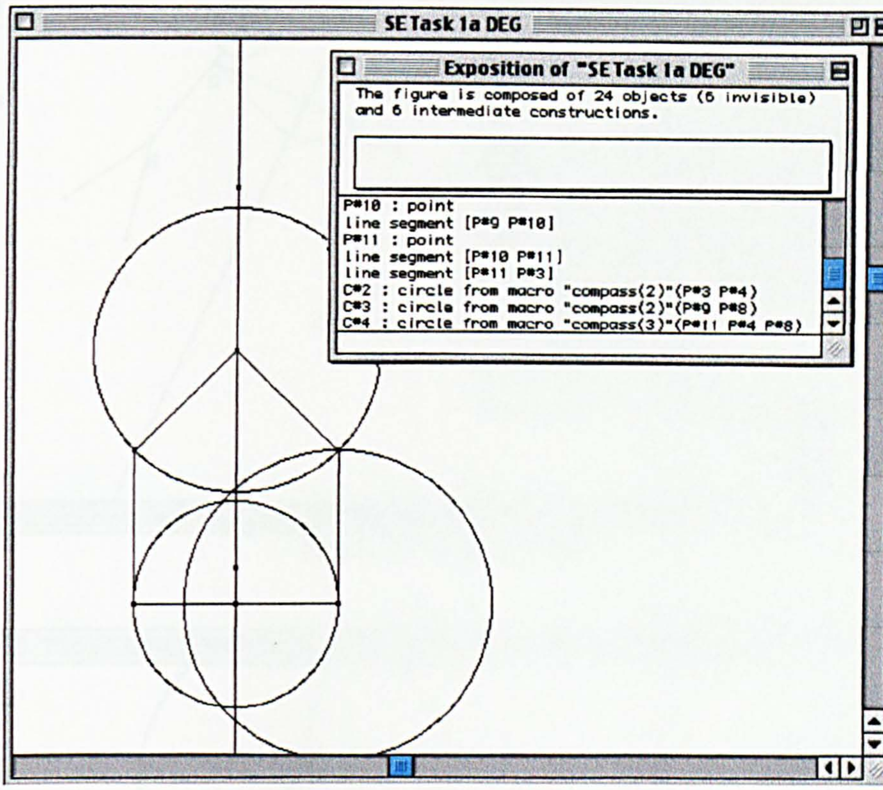


Figure A5.18:Task 1a DEG

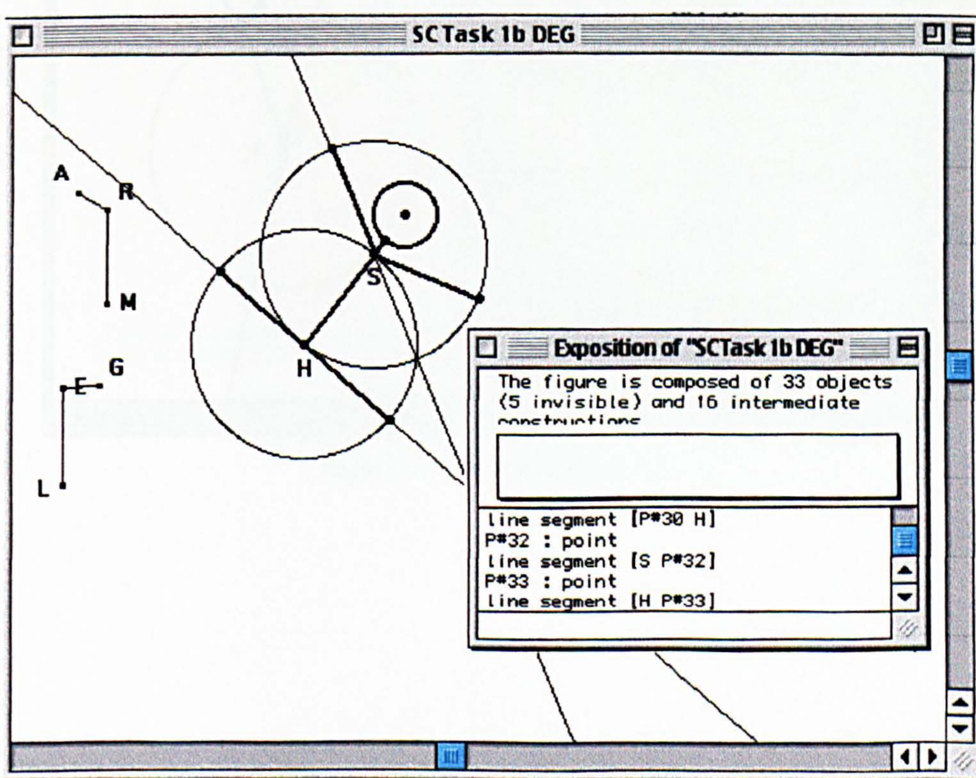


Figure A5.19:Task 1b DEG

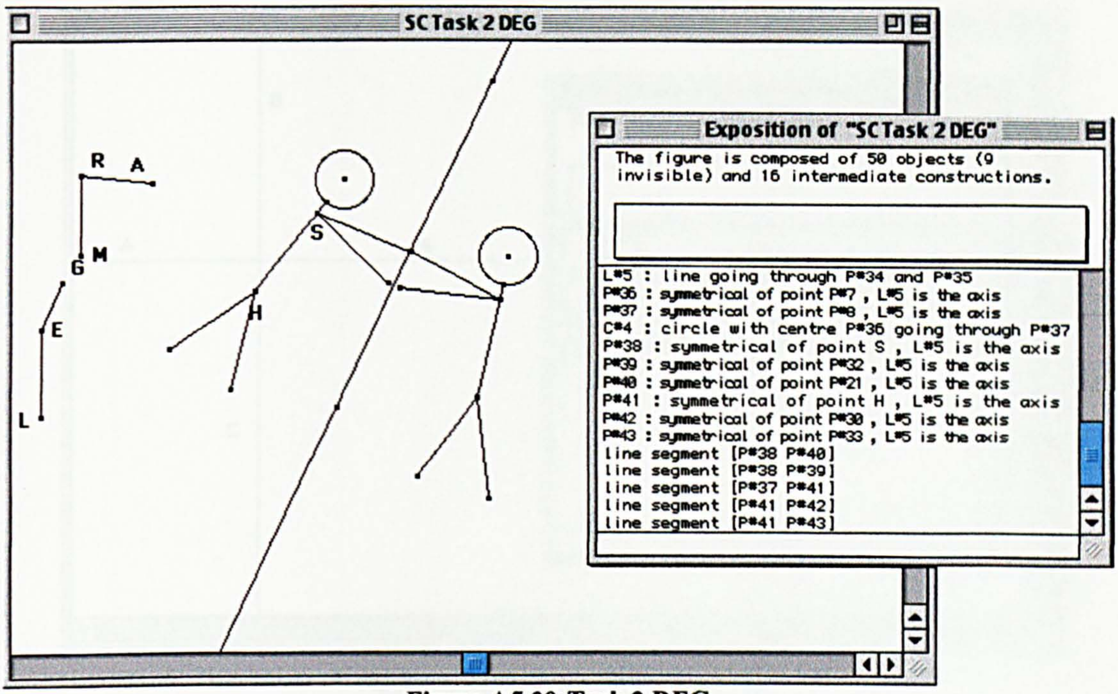


Figure A5.20:Task 2 DEG

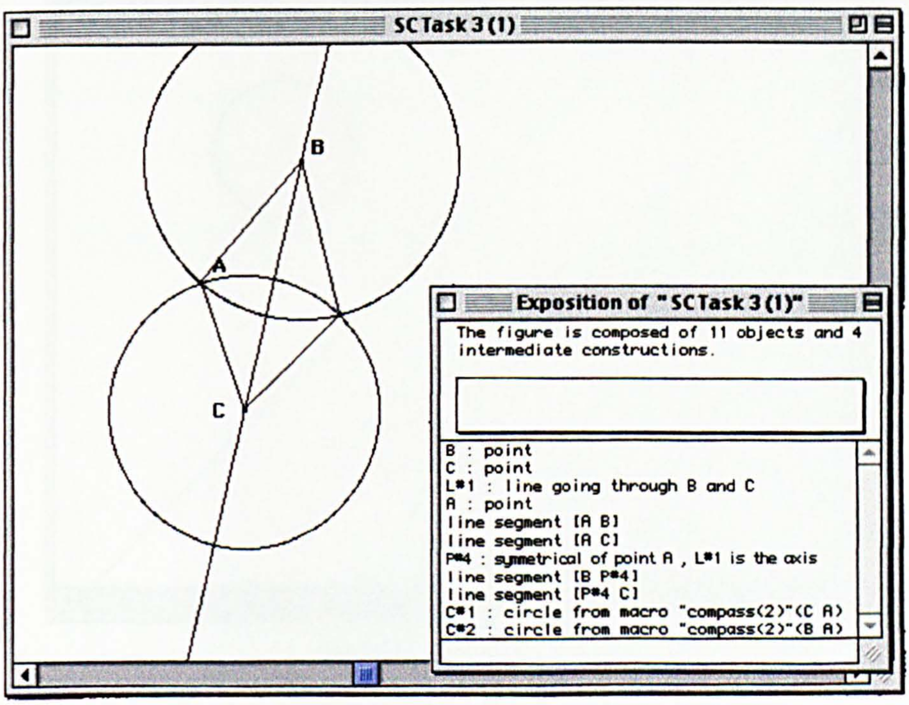


Figure A5.21:Task 3 (figure 1)

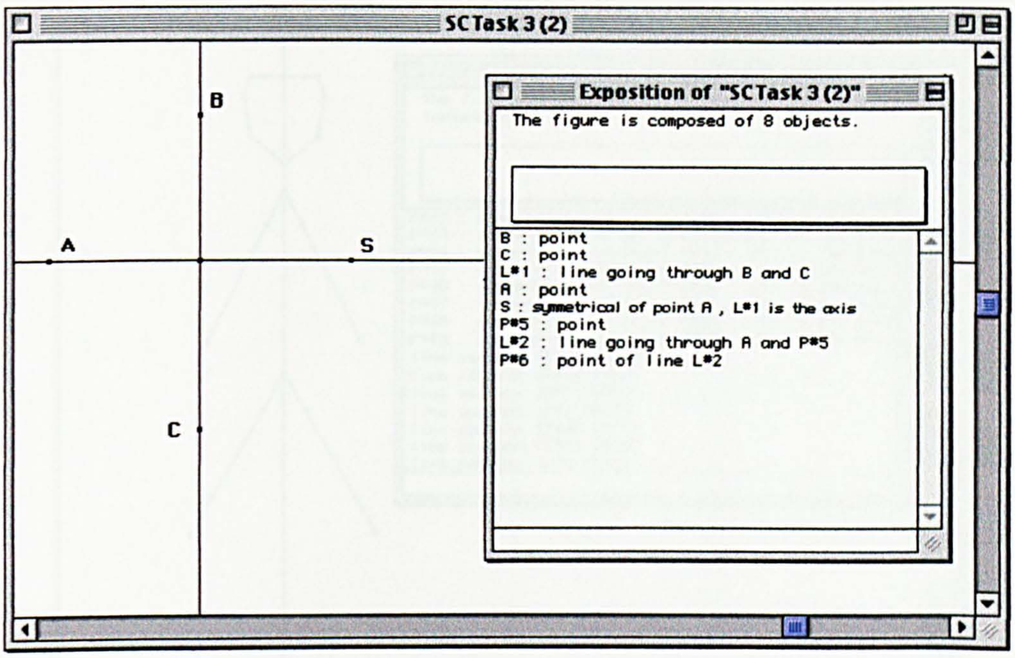


Figure A5.22:Task 3 DEG (figure 2)

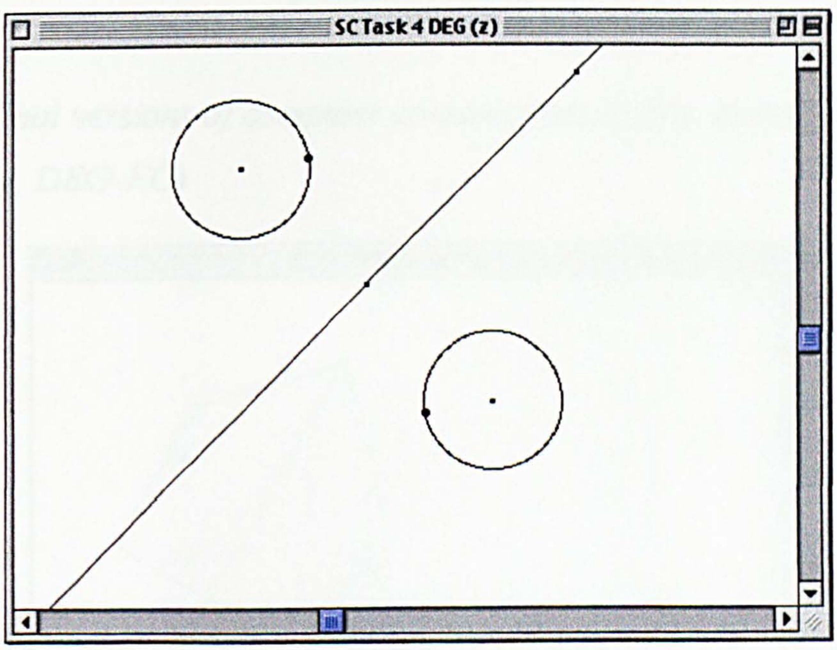


Figure A5.23:Task 4 DEG

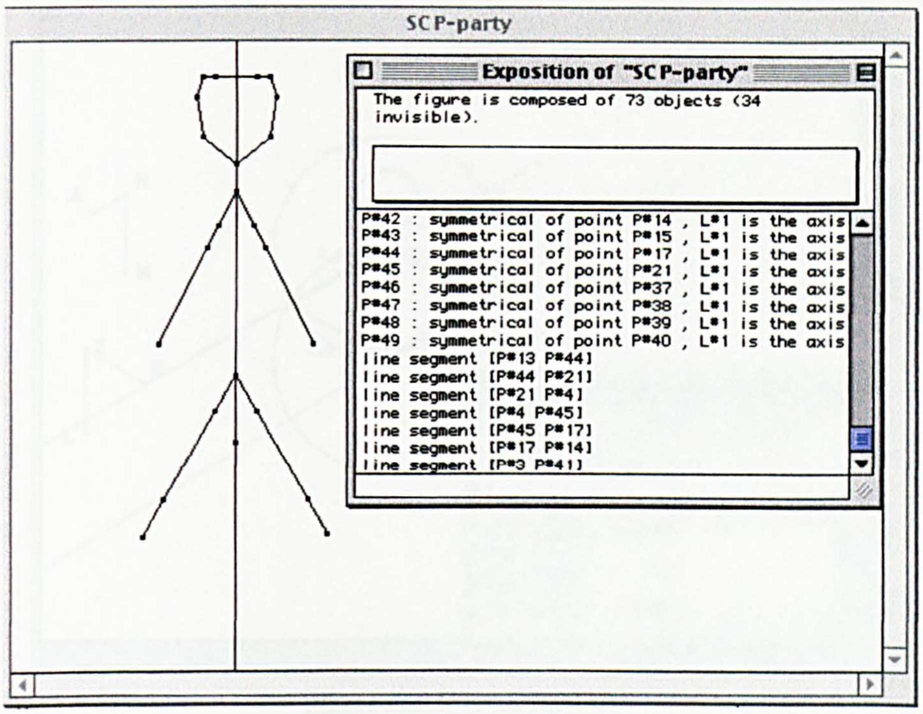


Figure A5.24:Task 5 DEG

A5.1.4 Final versions of computer constructions of Sita and Anju (type I-II pairing, DEG-FO)

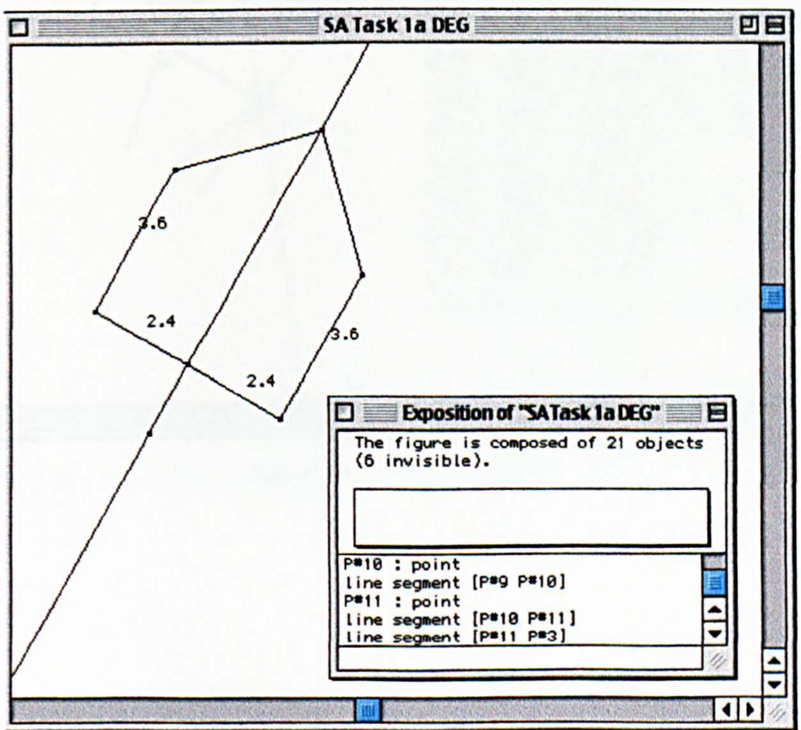


Figure A5.25:Task 1a DEG

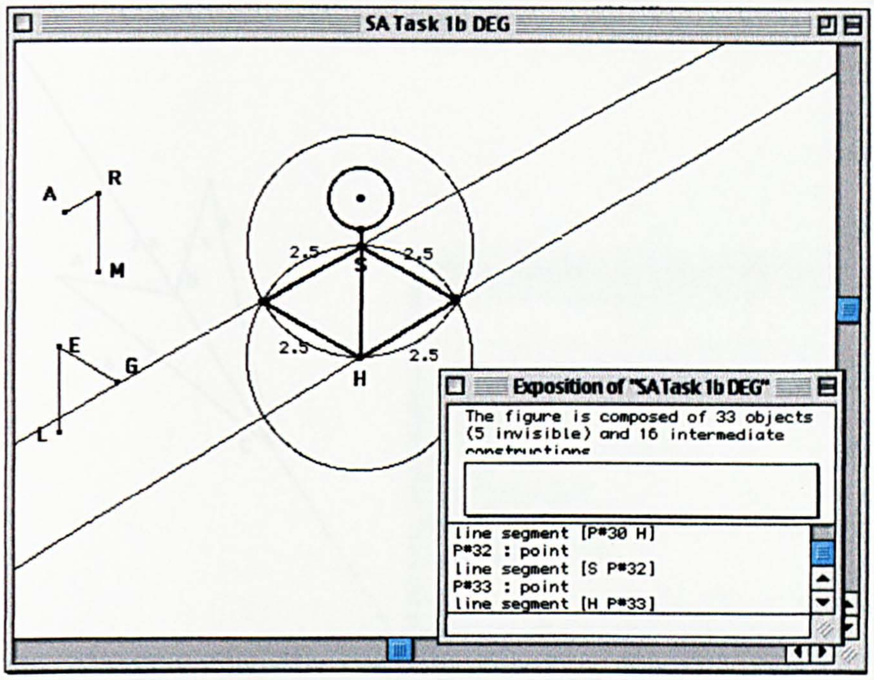


Figure A5.26:Task 1b DEG

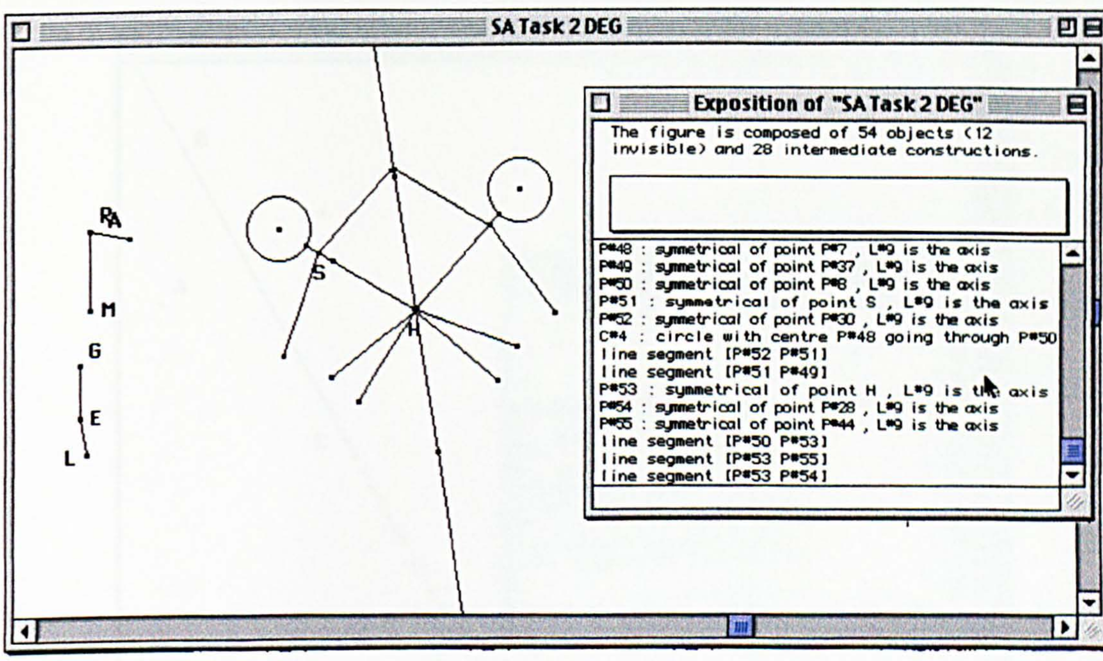


Figure A5.27:Task 2 DEG

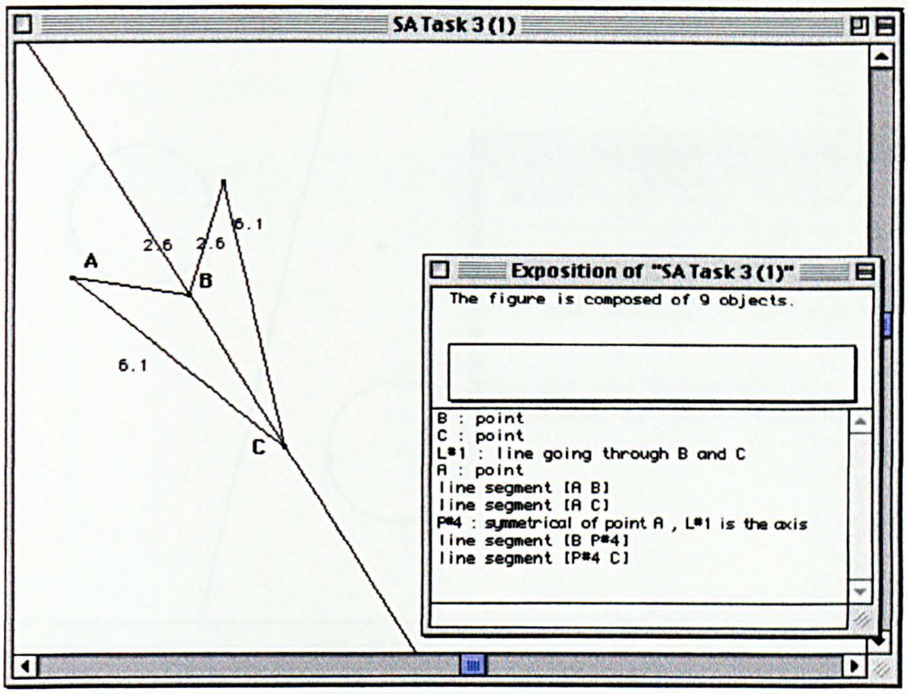


Figure A5.28:Task 3 (figure 1)

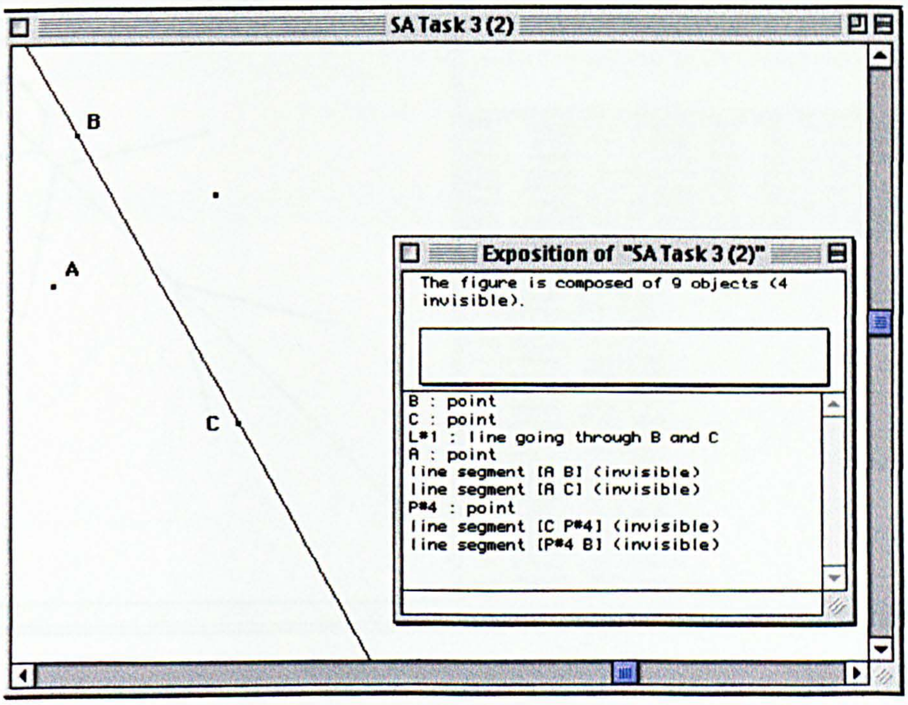


Figure A5.29:Task 3 DEG (figure 2)

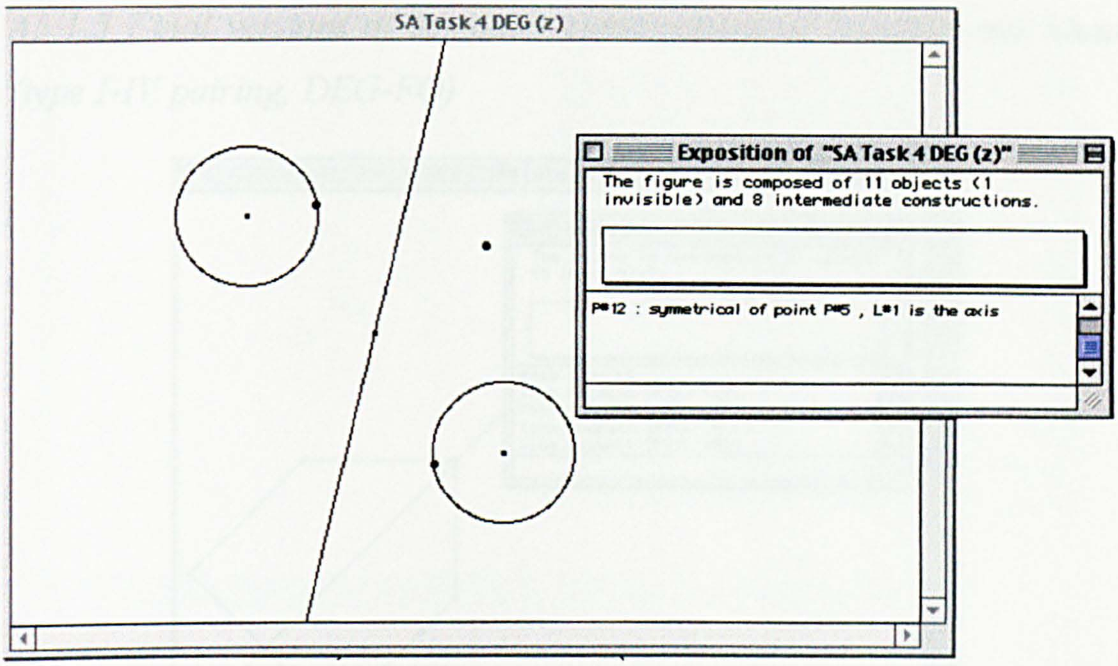


Figure A5.30:Task 4 DEG

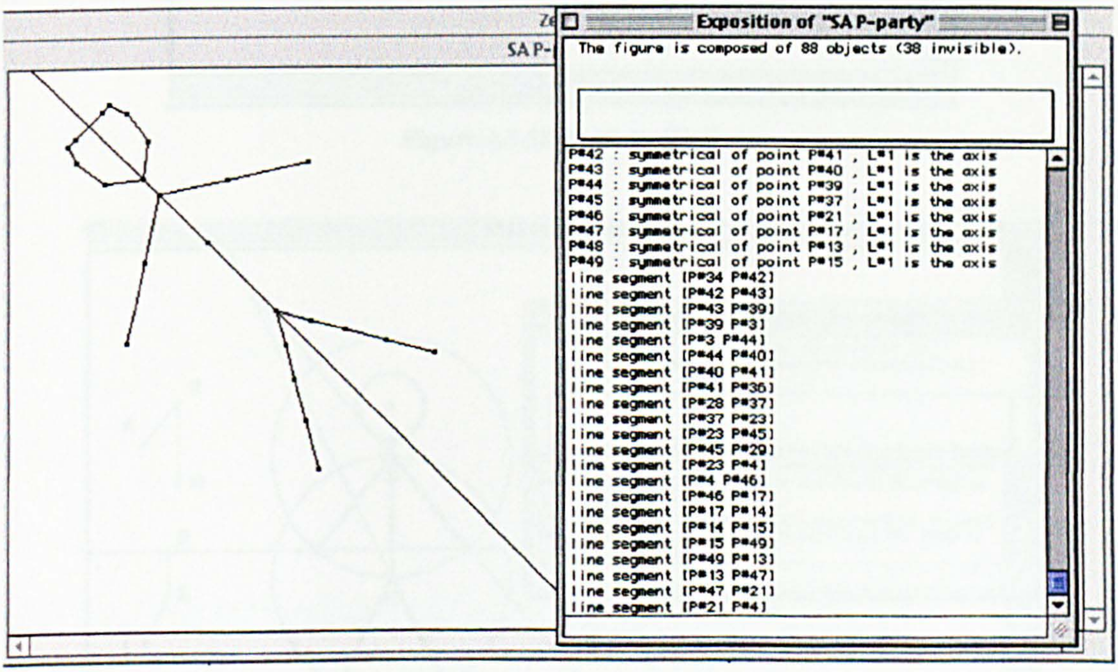


Figure A5.31:Task 5 DEG

A5.1.5 Final versions of computer constructions of Rebekka and Maia (type I-IV pairing, DEG-FO)

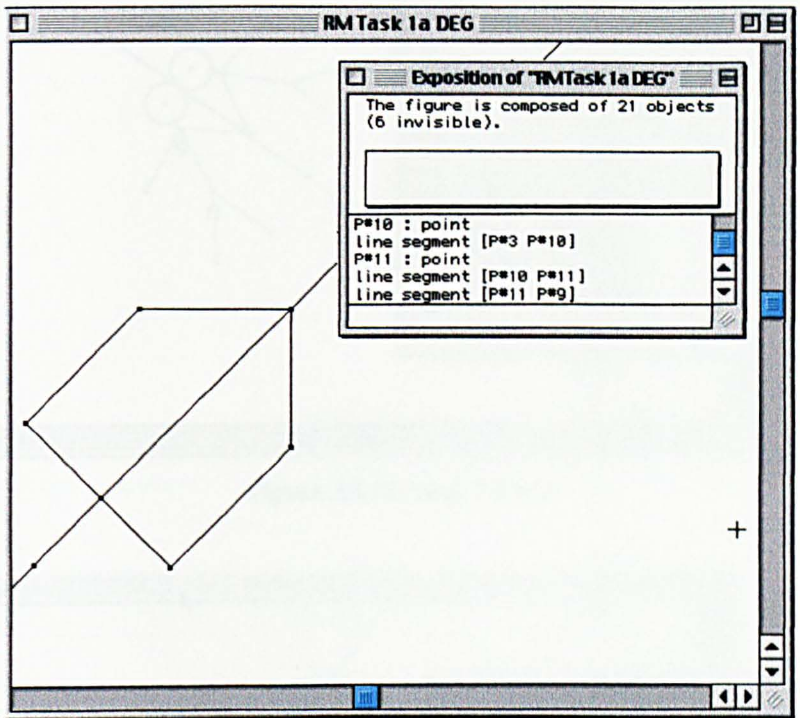


Figure A5.32:Task 1a DEG

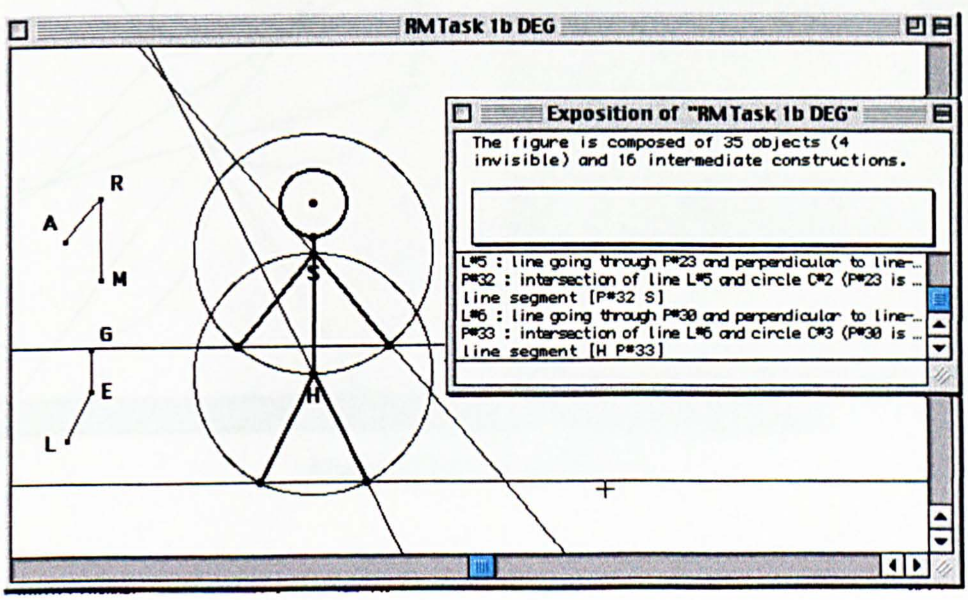


Figure A5.33:Task 1b DEG

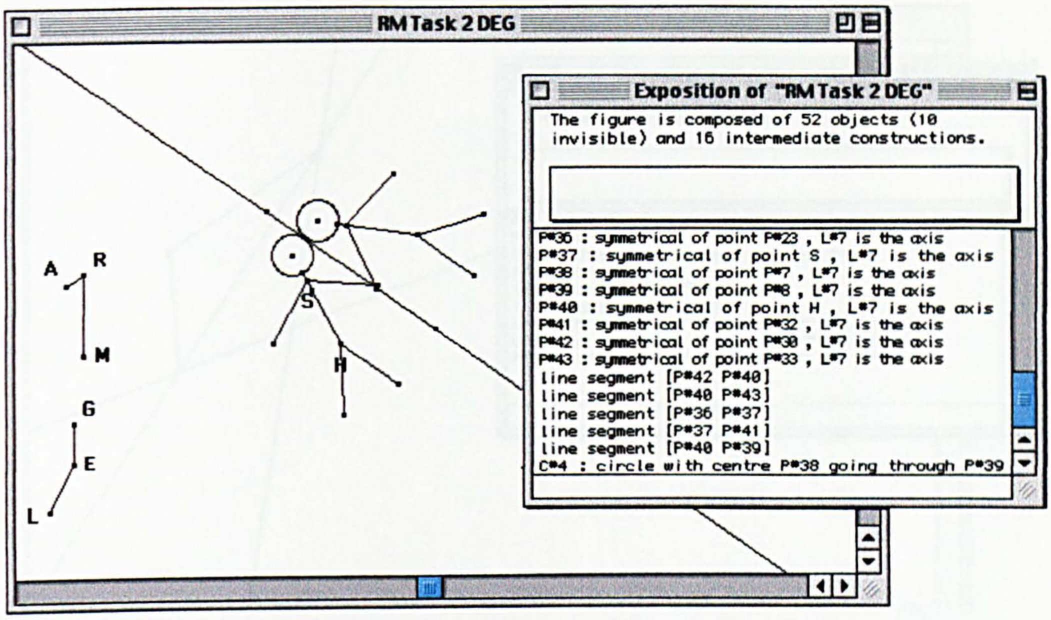


Figure A5.34:Task 2 DEG

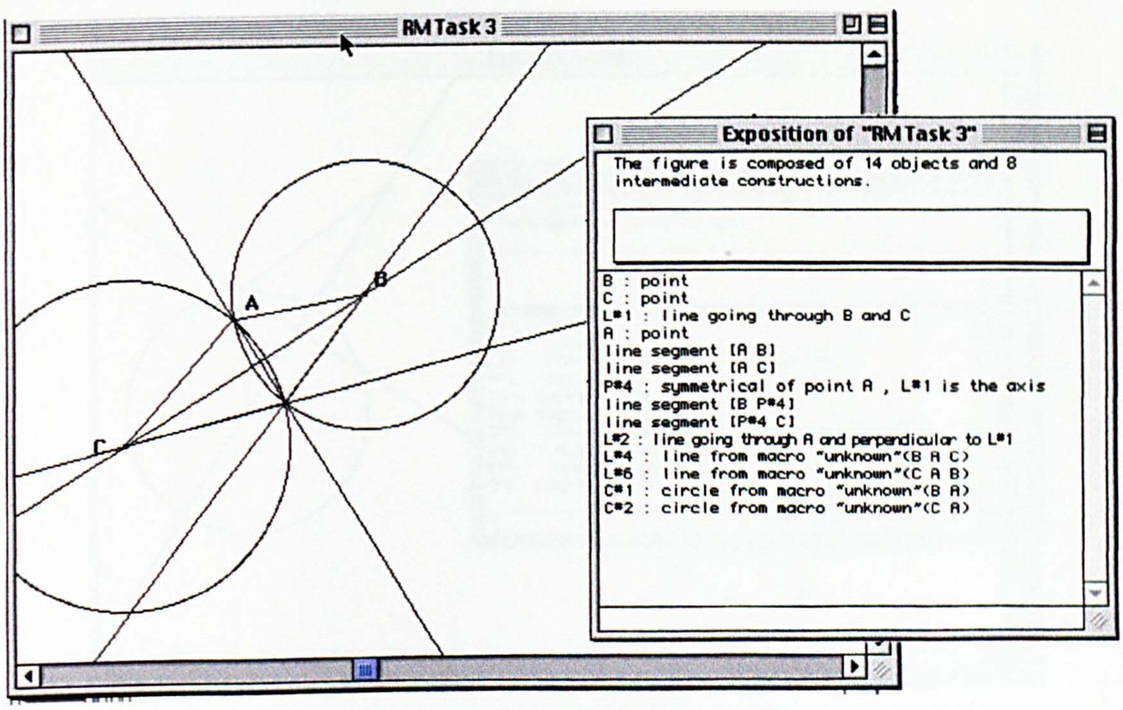


Figure A5.35:Task 3 (figure 1)

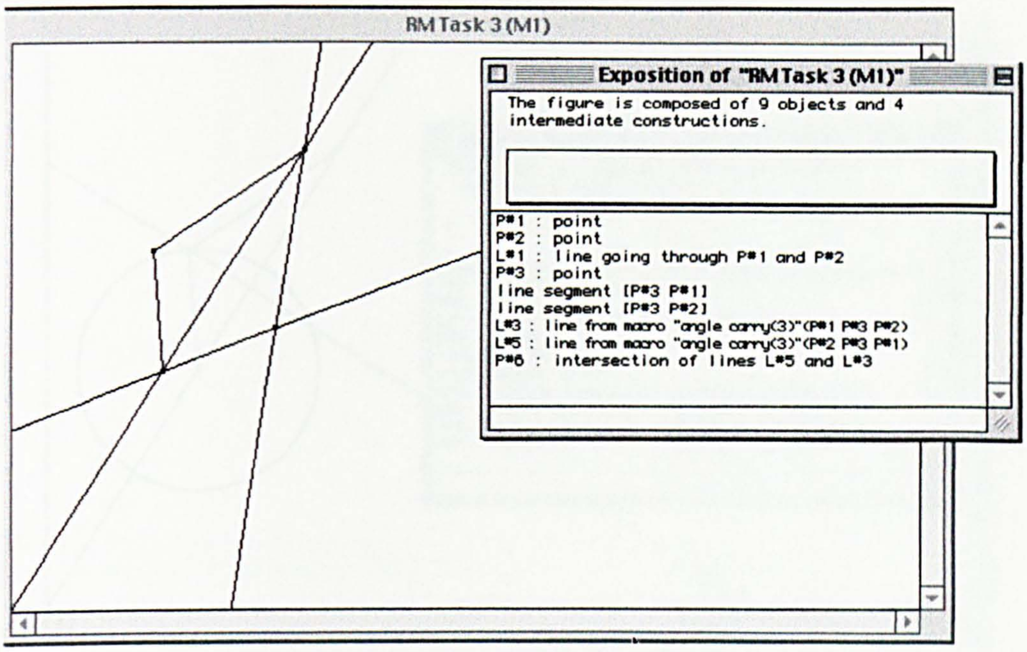


Figure A5.36: Task 3 DEG (figure 2)

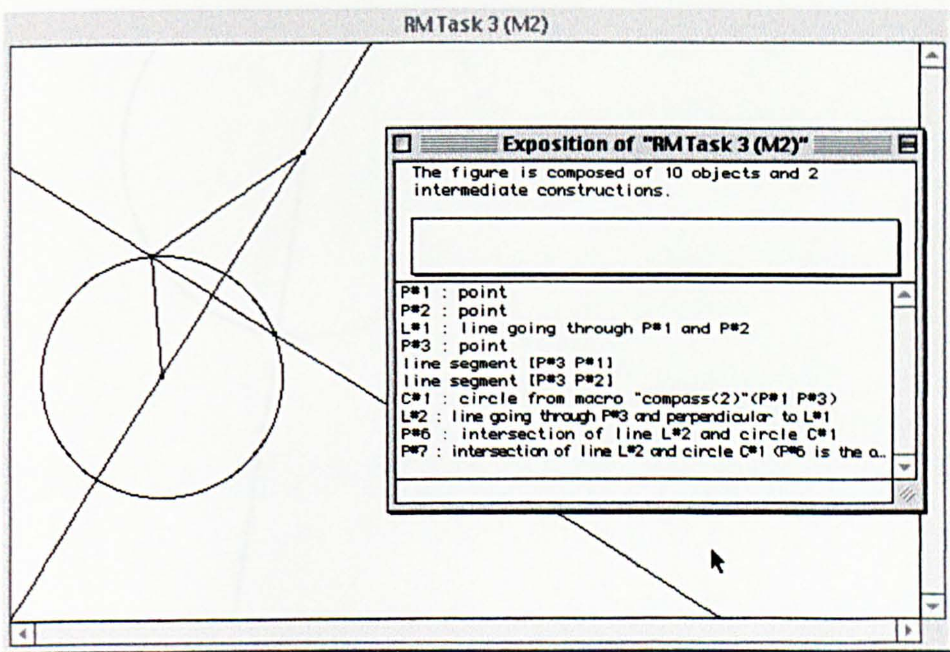


Figure A5.37: Task 3 DEG (figure 2)

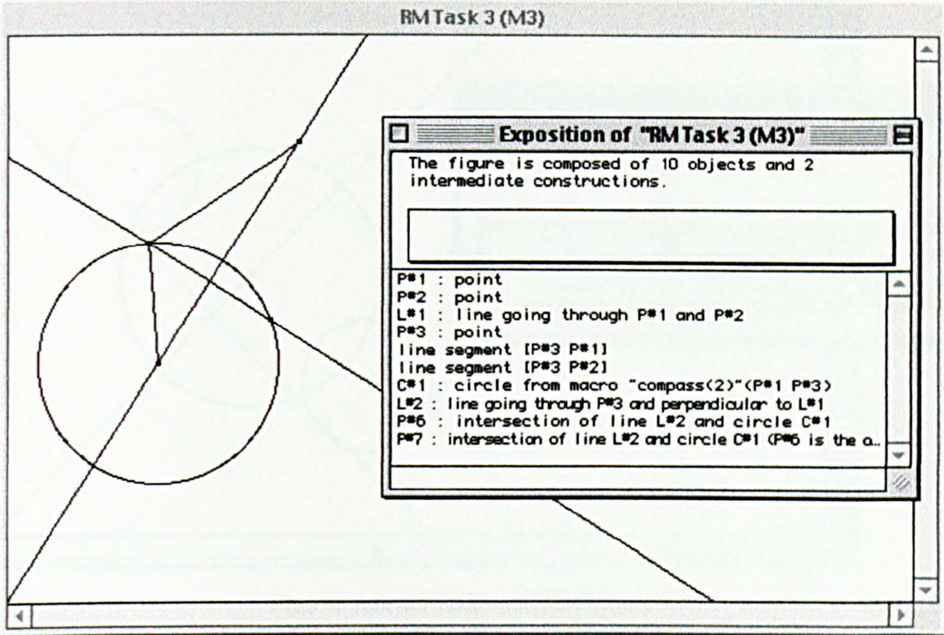


Figure A5.38:Task 3 DEG (figure 2)

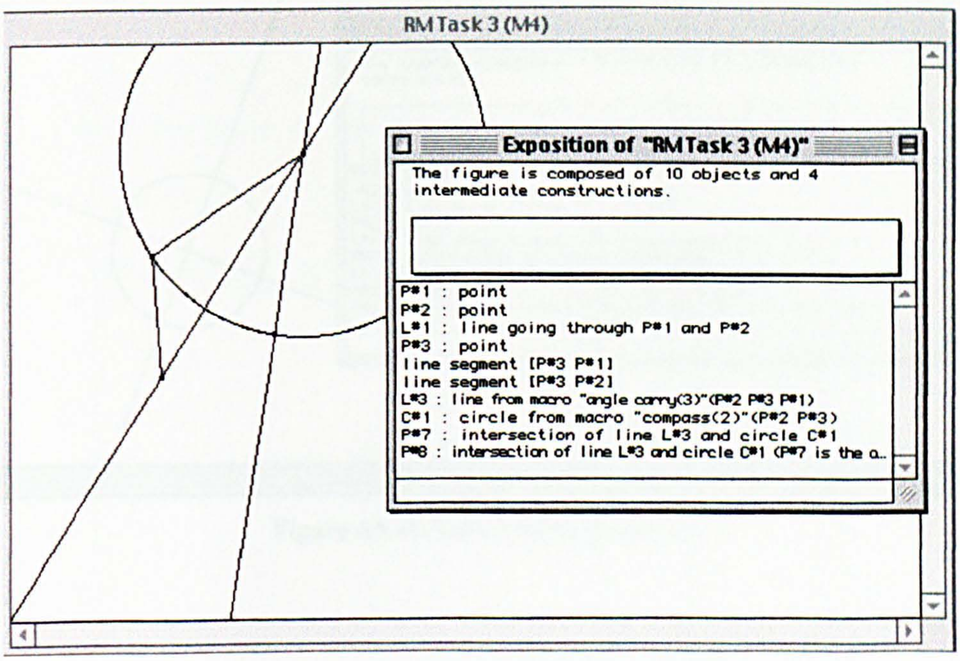


Figure A5.39:Task 3 DEG (figure 2)

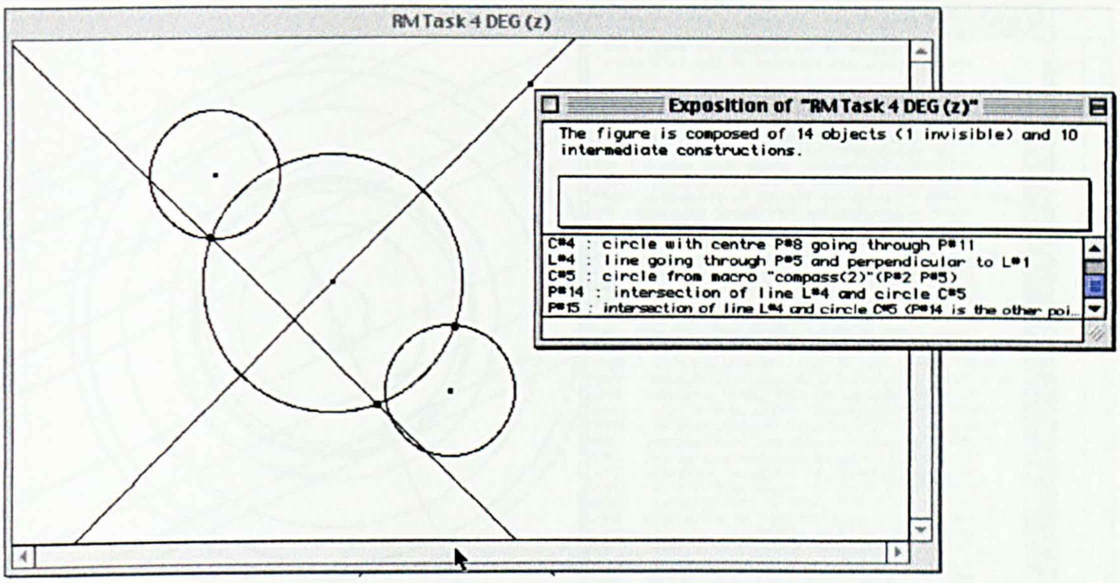


Figure A5.40:Task 4 DEG (figure 1)

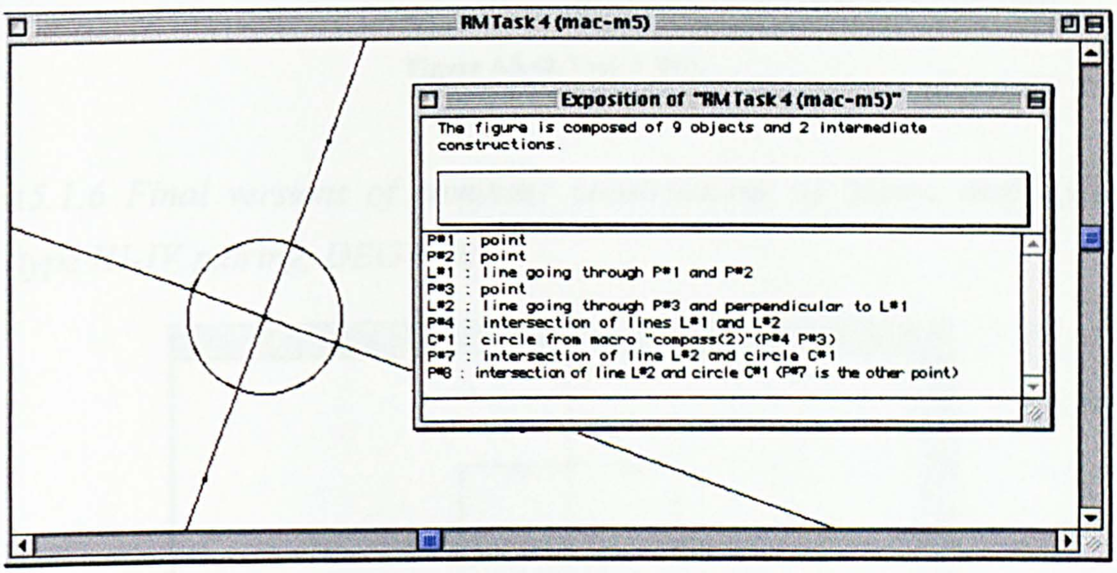


Figure A5.41:Task 4 DEG (figure 2)

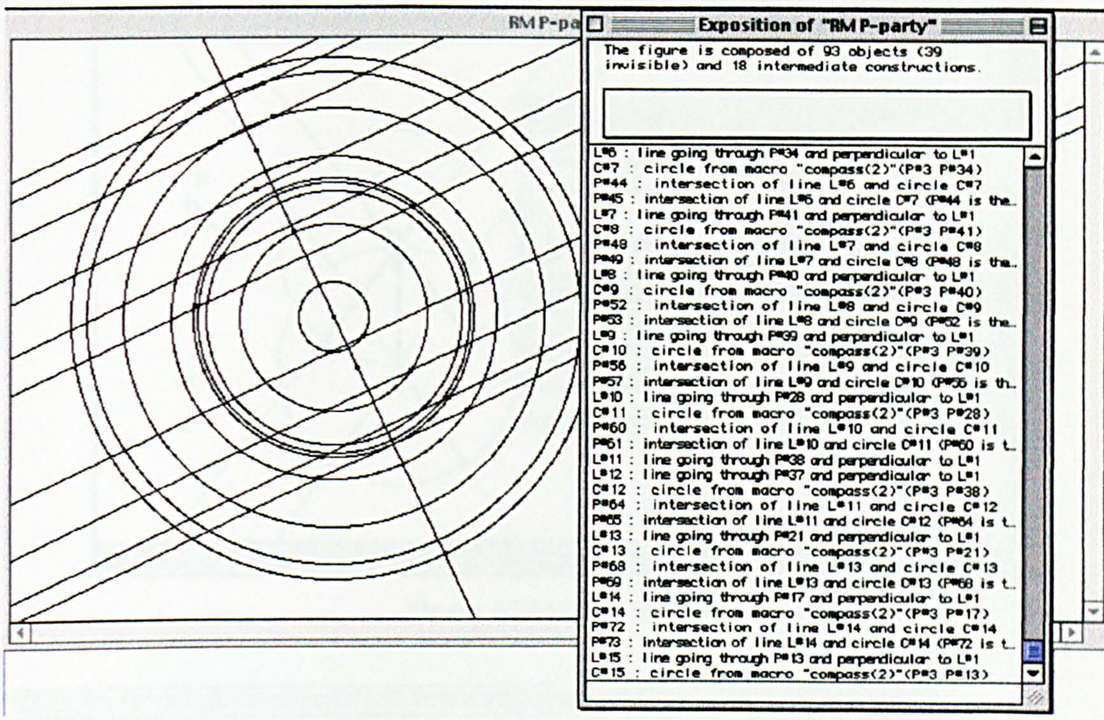


Figure A5.42:Task 5 DEG

A5.1.6 Final versions of computer constructions of Seema and Kylie (type III-IV pairing, DEG-FO)

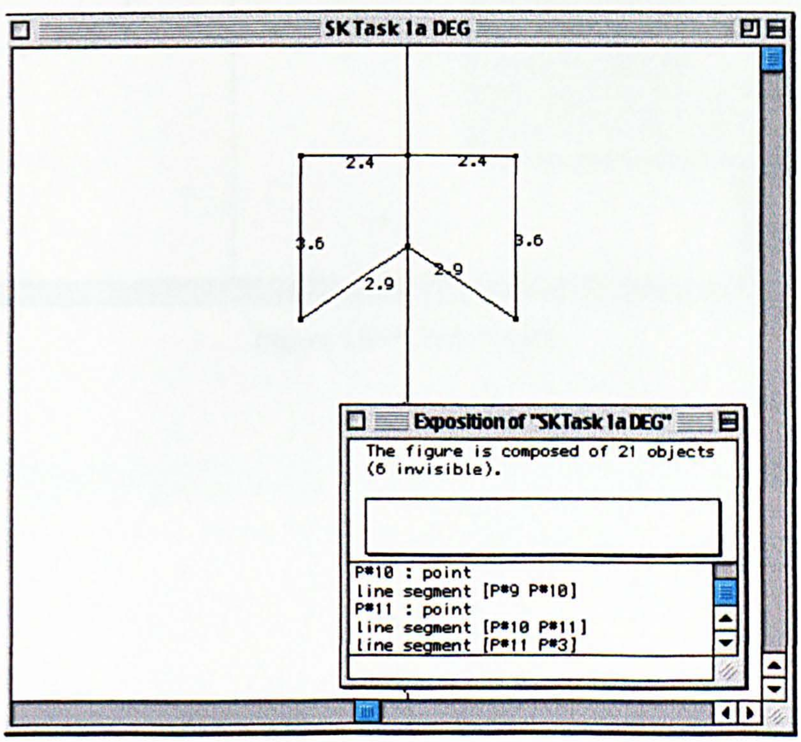


Figure A5.43:Task 1a DEG

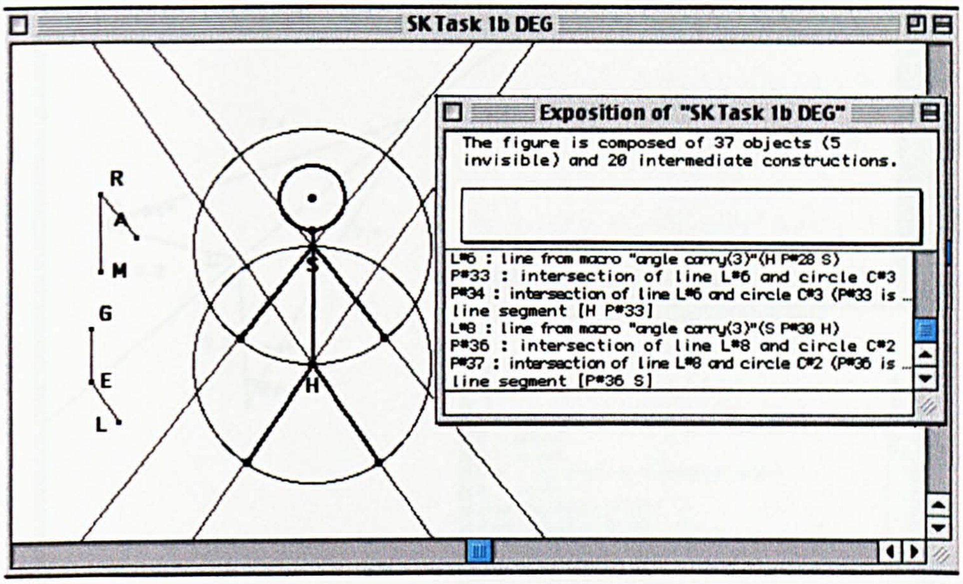


Figure A5.44:Task 1b DEG

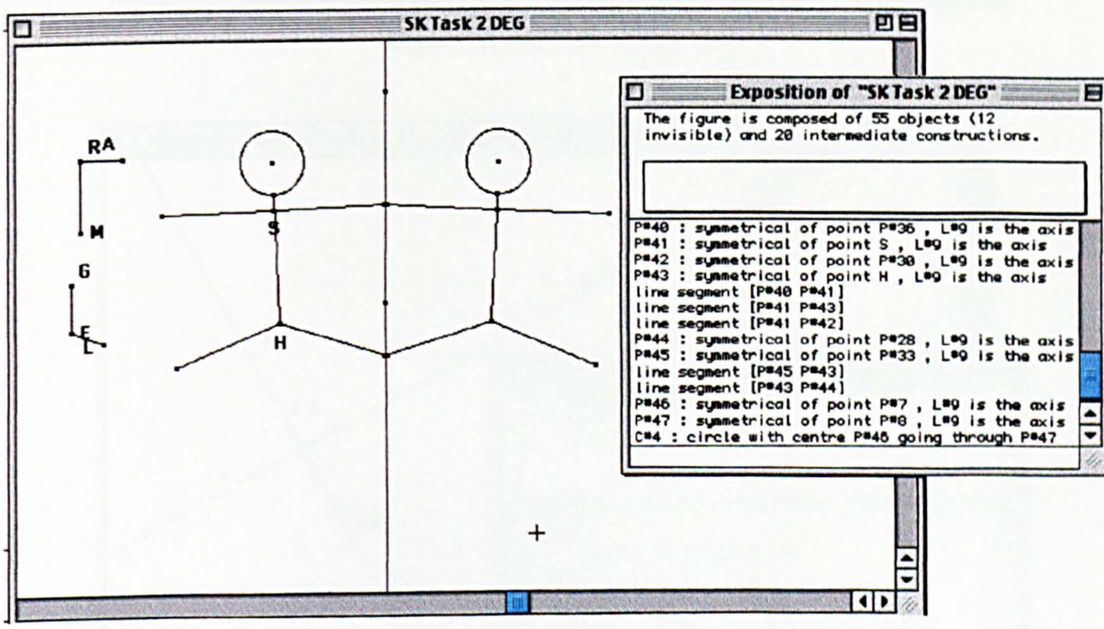


Figure A5.45:Task 2 DEG

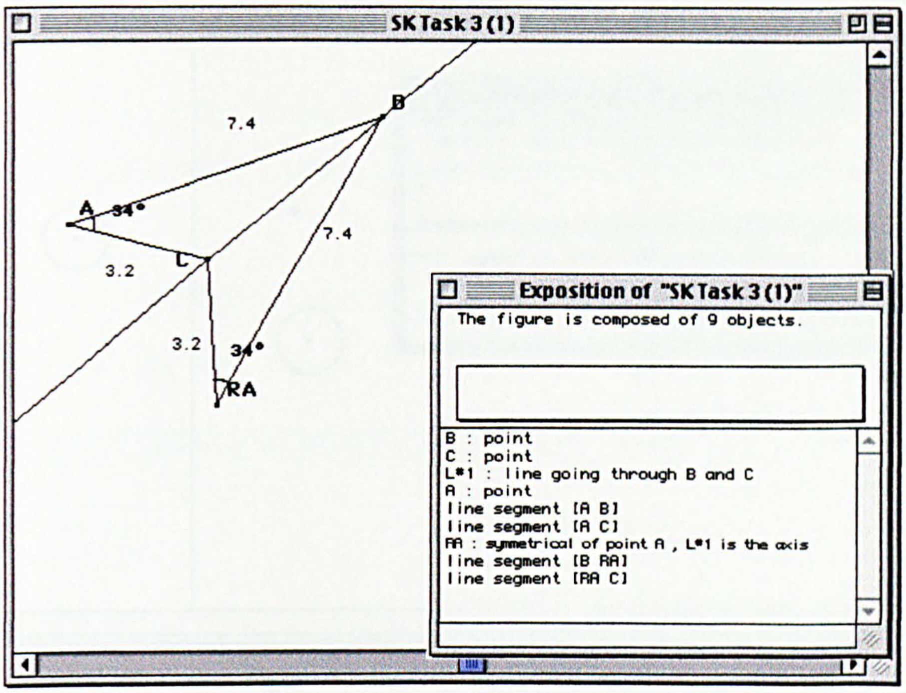


Figure A5.46:Task 3 (figure 1)

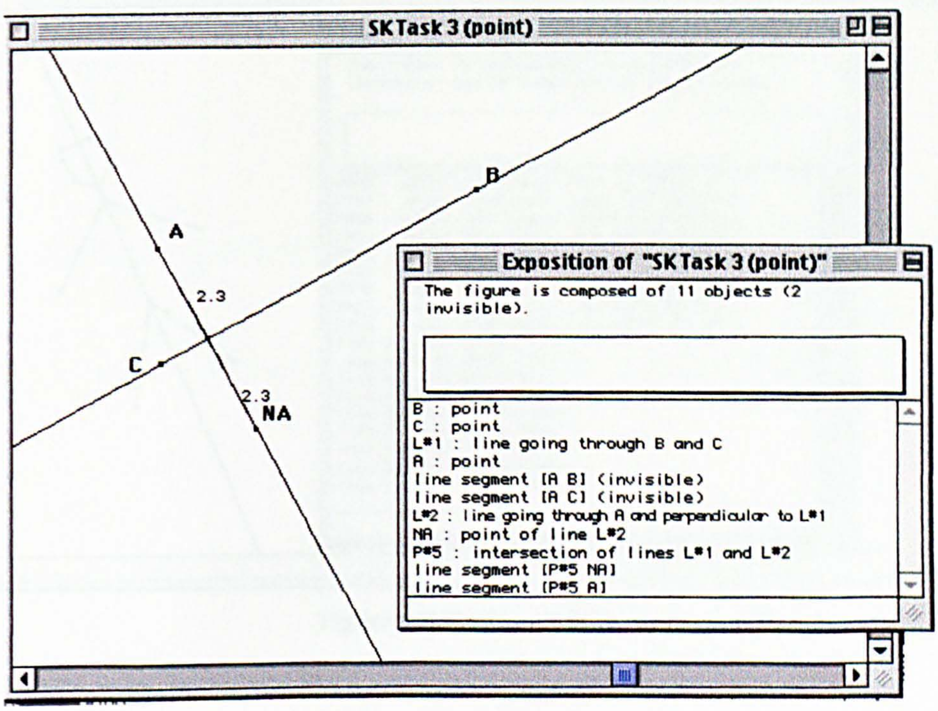


Figure A5.47:Task 3 DEG (figure 2)

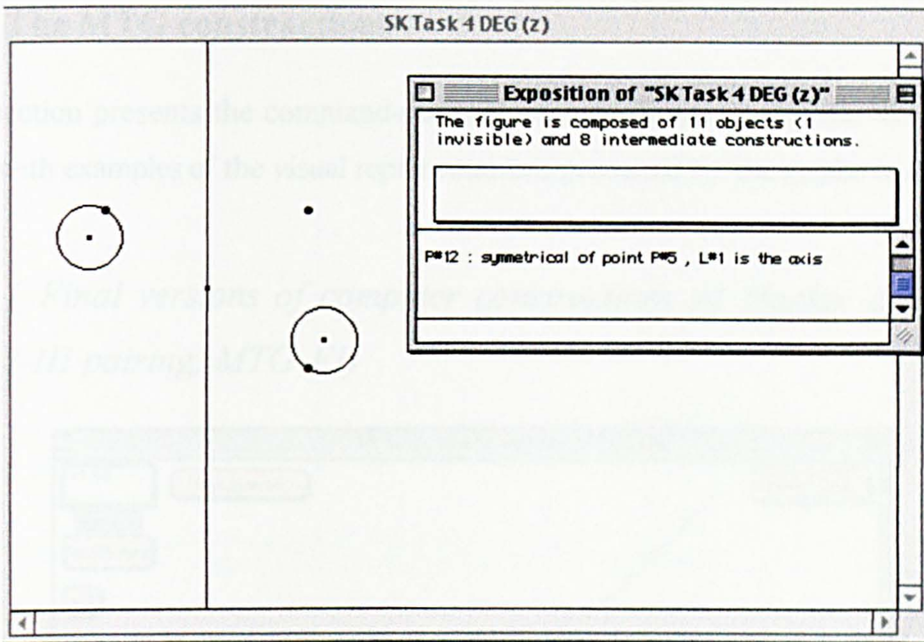


Figure A5.48:Task 4 DEG (figure 1)

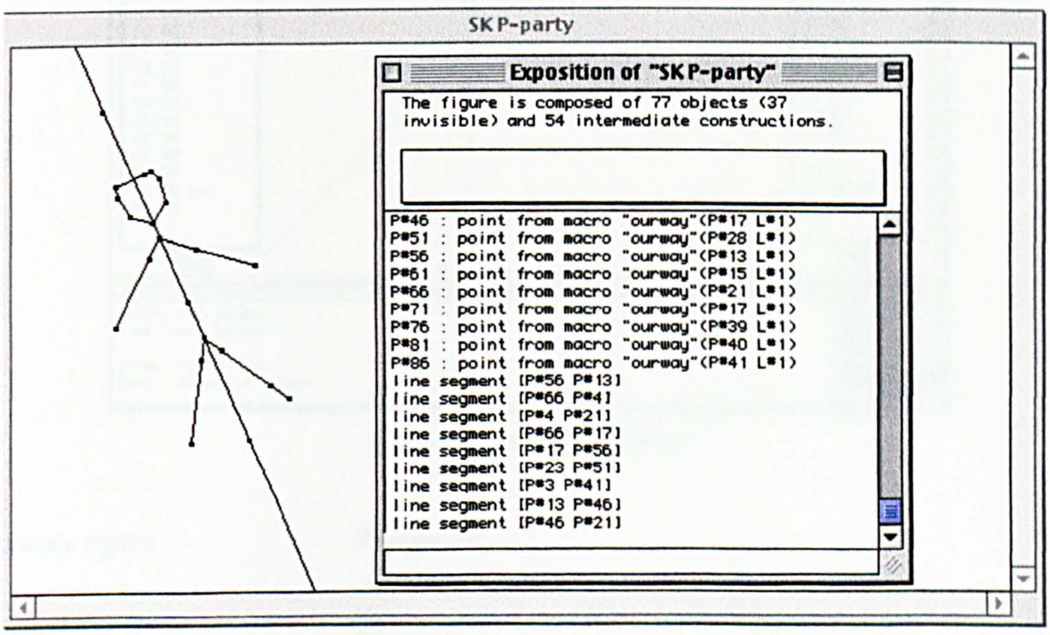


Figure A5.49:Task 5 DEG

A5.2 The MTG constructions

This section presents the command-sets and procedures written by the MTG pairs, along with examples of the visual representations produced by the symbolic code.

A5.2.1 Final versions of computer constructions of Hadley and Lorna (type I-III pairing, MTG-FI)

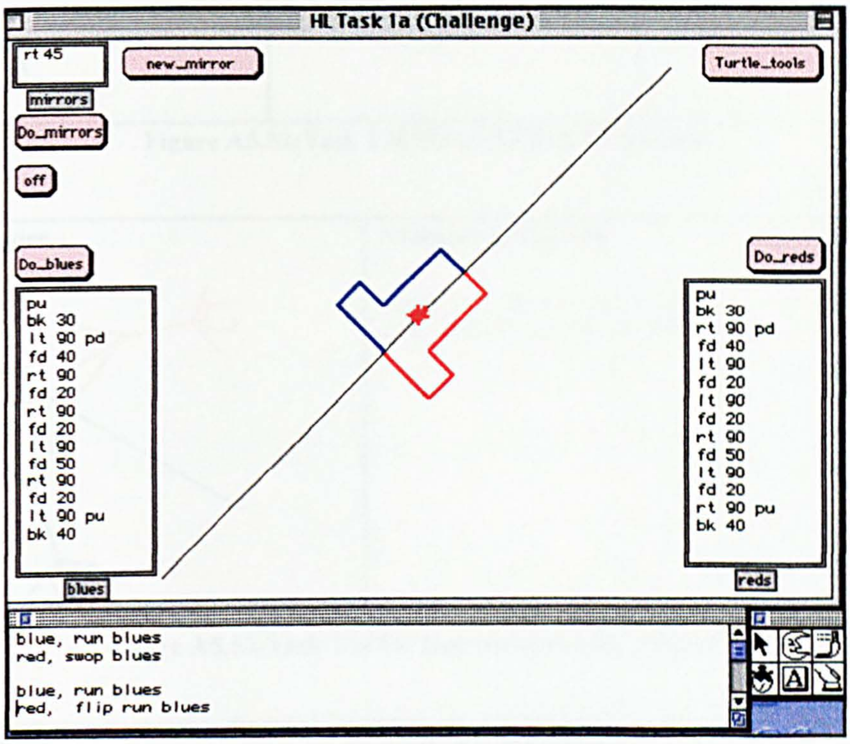


Figure A5.50:Task 1a MTG

Example figure

Procedures

```
to per :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

```
to son :sh :hip
pu
fd 80 pd
rt 90 fd 10 rt 90
fd 20 rt 90 fd 10
lt 90
fd 10
lt :sh
fd 30 bk 30
rt :sh
fd 50
lt :hip
fd 50 bk 50
rt :hip
rt 180
end
```

Figure A5.51:Task 1b MTG

Example figure	Example commands	Procedures

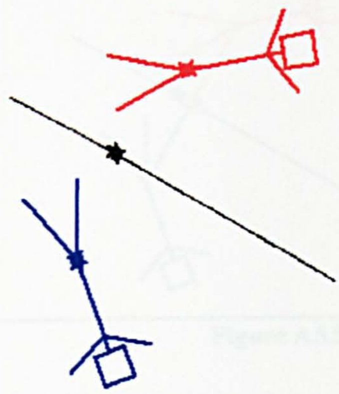
	`blue, lh 100 70 40`
	`pd per 120 20 son 120 20`
	`red, swop [lh 100 70 40]`
	`pd per 120 20 son 120 20`
		`to lh :slip :slap :slop`
		`lt :slip fd :slap`
		`lt :slop`
		`end`

Figure A5.52:Task 2 MTG (reflecting the person)

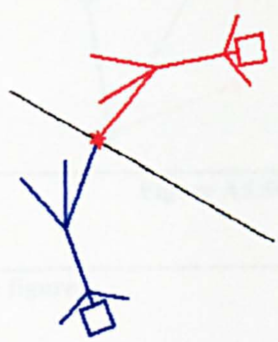
Example figure	Example commands
	`blue, rt 40 bk 70 rt 100` `red, lt 40 bk 70 lt 100`

Figure A5.53:Task 2 MTG (reuniting turtles, method 1)

Example figure	Example commands

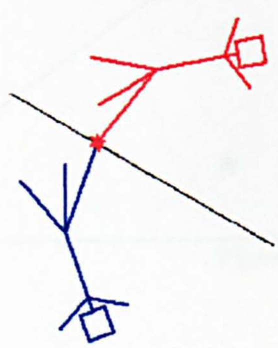
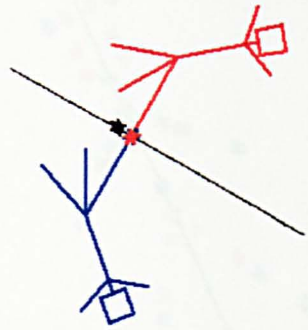
	`blue, show towards "mirror`
	`lt 140`
	`blue, show distance "mirror`
	`fd 70`
	`blue, show toheading "mirror`
	`lt 80`
	`red, rt 140 fd 70 rt 80`

Figure A5.54:Task 2 MTG (reuniting turtles, method 2)

Example figure

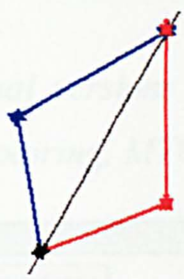


Example commands

```
blue, show towards "red
lt 130
blue, show distance "red
fd 79
red, rt 130
fd 79
```

Figure A5.55:Task 2 MTG (reuniting turtles, method 3)

Example figure

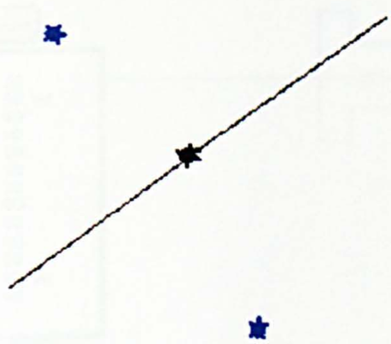


Example commands

```
blue, pd lt 40 fd 100
red, pd rt 40 fd 100
blue, rt 70
red, lt 70
blue, meet "red
red, meet "blue
remember distance "red
blue, fd :m1 bk :m1
red, :m1 bk :m1
```

Figure A5.56:Task 3 MTG (symmetrical quadrilateral)

Example figure



Procedure

```
to lhimage
hatchhere
remember towards "mirror
run :m1 rt 180
remember distance "mirror
remember toheading "mirror
run :m3 run :m3
fd :m2
lt 180 run :m1
end
```

Figure A5.57:Task 3 MTG (image turtle)

Figure	Commands
	everyone [lhimage forget]

Figure A5.58:Task 5 MTG

A5.2.2 Final versions of computer constructions of Lizzie and Aimie (type I-IV pairing, MTG-FI)

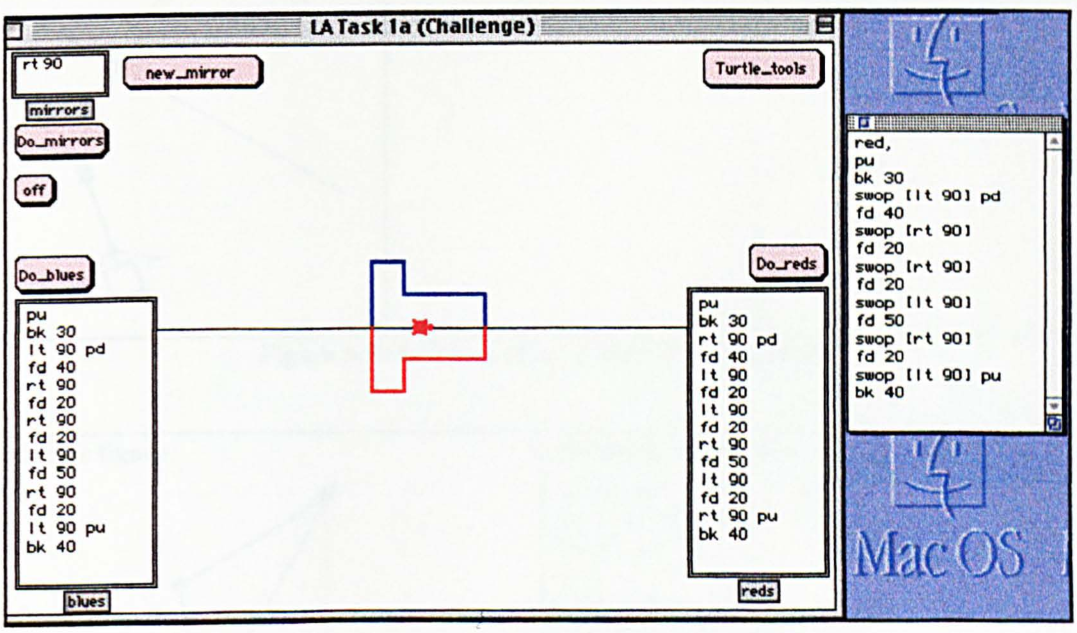


Figure A5.59:Task 1a MTG

Example figure

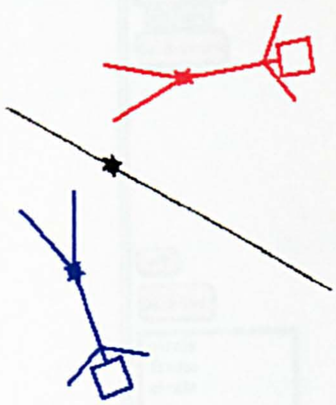
Procedures

```
to si :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

```
to mon :sh :hip
pu
fd 80 pd
rt 90 fd 10 rt 90
fd 20 rt 90 fd 10
lt 90
fd 10
lt :sh
fd 30 bk 30
rt :sh
fd 50
lt :hip
fd 50 bk 50
rt :hip
rt 180
end
```

Figure A5.60:Task 1b MTG

Example figure



Example commands

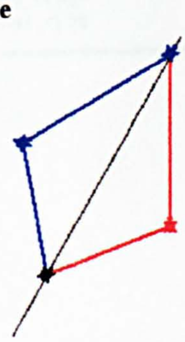
```
blue, lt 100 fd 70 lt 40
pd simon 120 20
blue, lt 100 fd 70 lt 40
pd simon 120 20
```

Procedures

```
to simon :sh :hip
si :sh :hip
mon :sh :hip
end
```

Figure A5.61:Task 2 MTG (reflecting the person)

Example figure

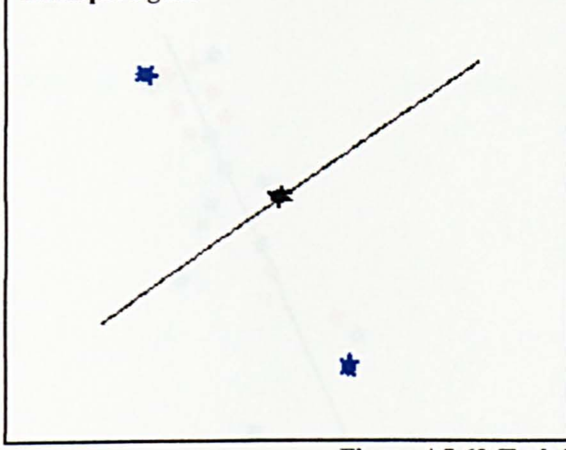


Example commands

```
blue, pd lt 40 fd 100
red, pd rt 40 fd 100
blue, rt 70
red, lt 70
blue, meet "red
remember distance "red
remember toheading "mirror
lt 30
blue, fd 107 bk 107
red, fd 107 bk 107
```

Figure A5.62:Task 3 MTG (symmetrical quadrilateral)

Example figure



Example commands

```
meet "mirror
remember toheading "mirror
lt 45
remember towards "blue
rt 135
remember distance "blue
fd 142
rt 180
```

Figure A5.63:Task 3 MTG (image turtle)

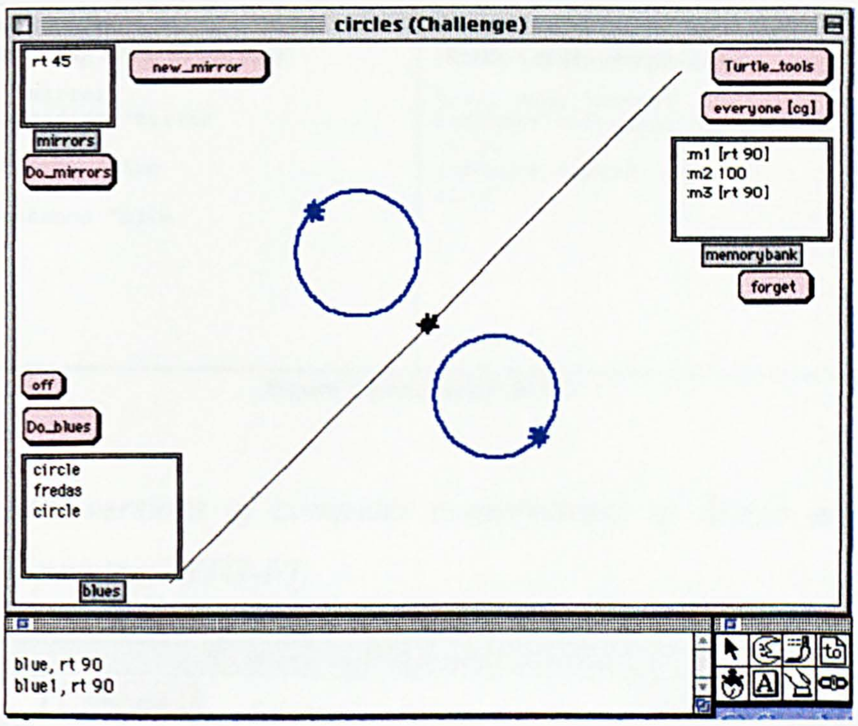


Figure A5.64:Task 4 MTG

Figure	**Example commands**
	(turtle facing towards mirror)
	`red, (lt 45) meet "mirror`
	`remember toheading "mirror`
	`rt 45`
	`remember towards "red`
	`lt 135`
	`remember distance "red`
	`fd 21`
	`rt 180`
	Commands for mirror turtle
	`mirror, hatchhere`

Example commands
(with bottom facing towards mirror)
`blue, meet "mirror`
`remember toheading "mirror`
`rt 30`
`remember towards "blue`
`rt 30`
`remember distance "blue`
`fd 51`
	Example commands
	(turtle s along mirror line)
	`gray, meet "mirror`
	`remember toheading "mirror`
	`rt 56`
	`remember towards "gray`
	`rt 56`

Figure A5.65:Task 5 MTG

A5.2.3 Final versions of computer constructions of Alissa and Helen (type II-IV pairing, MTG-FI)

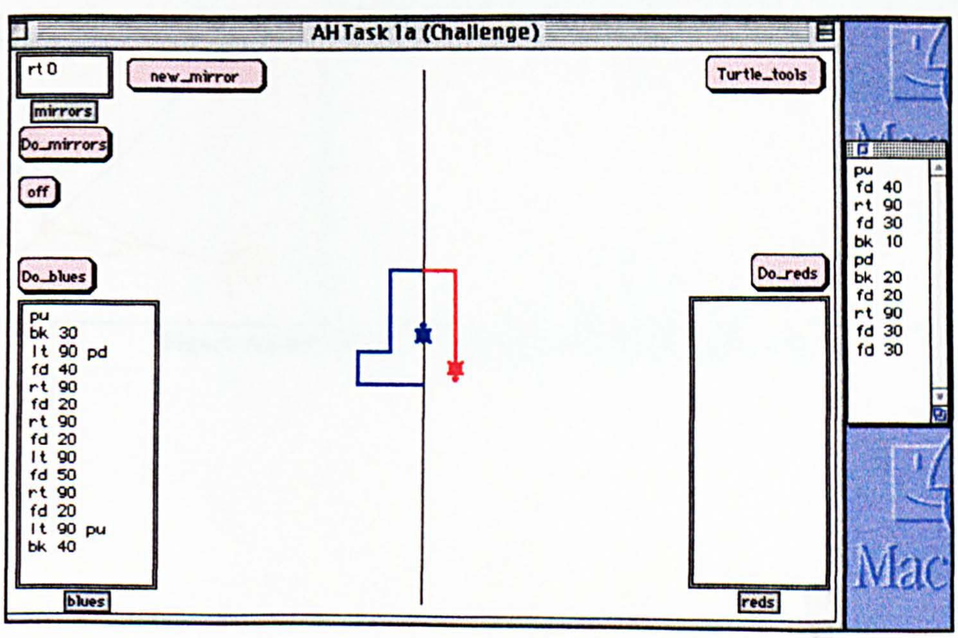


Figure A5.66:Task 1a MTG

Example figure

Procedures

```
to per :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

```
to aimee
pu
fd 80 pd
rt 90 fd 10 rt 90
fd 20 rt 90 fd 10
lt 90
fd 10
lt 45
fd 30 bk 30
rt 45
fd 50
lt 30
fd 50 bk 50
rt 30
rt 180
end
```

Figure A5.67:Task 1b MTG

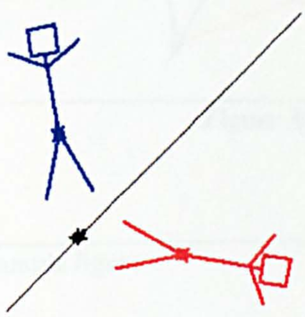
Example figure	Example commands	Procedures
	blue, alissa 55 80 pd si 120 20 mon 120 20 red, helen 55 80 pd mon 120 20 si 120 20	to alissa :t :f lt :t fd :f end to helen :t :f rt :t fd :f **end**

Figure A5.68:Task 2 MTG (reflecting the person)

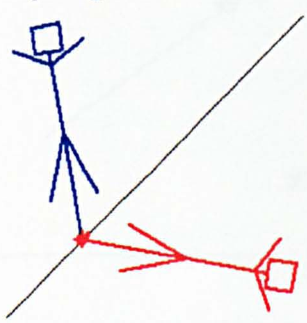
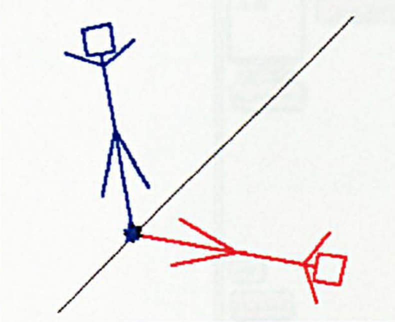
Example figure	Example commands
	**blue, bk 80 rt 55** **red, bk 80 lt 55**

Figure A5.69:Task 2 MTG (reuniting turtles, method 1)

Example figure

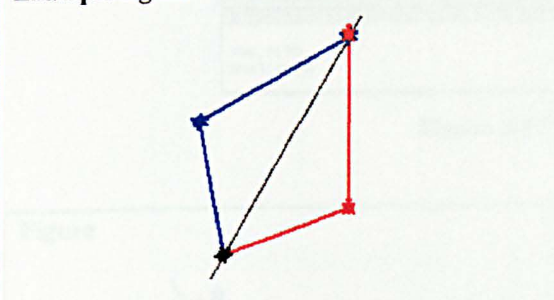


Example commands

```
blue, meet "red
show distance "blue
blue, bk 80
red, bk 80
```

Figure A5.70:Task 2 MTG (reuniting turtles, method 2)

Example figure

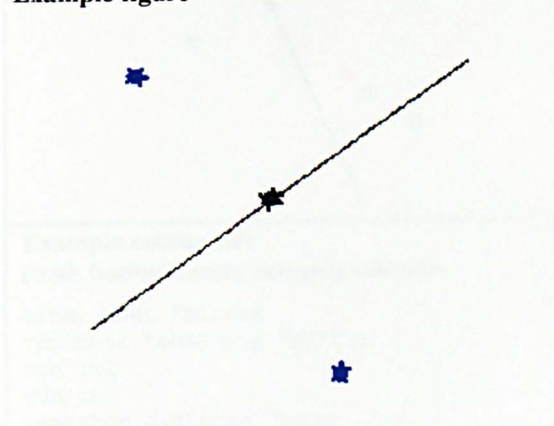


Example commands

```
blue, lt 40 fd 100 rt 70
meet "mirror
remember distance "blue
blue, pd fd 107 bk 107
red, rt 40 fd 100 lt 70
meet "mirror
red, pd fd 107 bk 107
```

Figure A5.71:Task 3 MTG (symmetrical quadrilateral)

Example figure



Procedure

```
to bluey
blue, meet "mirror
remember toheading "mirror
run :m1
run :m1
remember distance "blue
bk :m2
end
```

Figure A5.72:Task 3 MTG (image turtle)

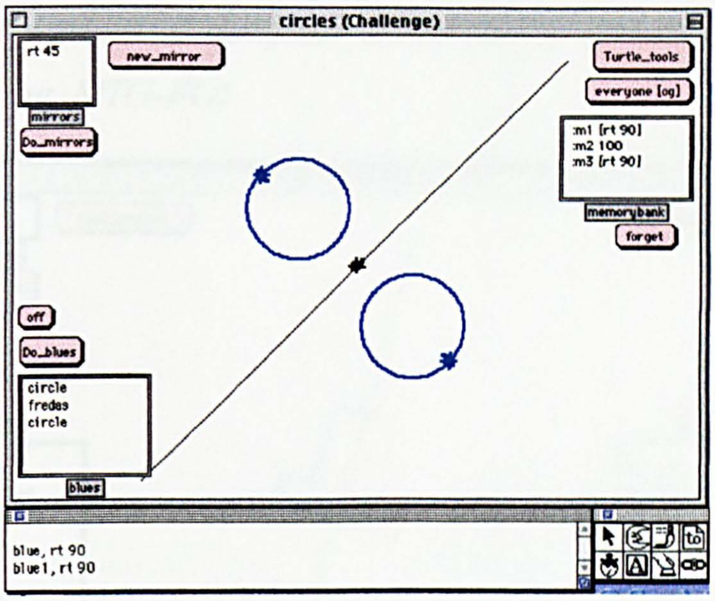


Figure A5.73:Task 4 MTG

Figure	**Example commands** **(turtle facing towards mirror)**
	pink, meet "mirror remember toheading "mirror run :m1 run :m1 remember distance "pink bk :m2
Example commands **(with bottom facing towards mirror)** blue, meet "mirror remember toheading "mirror run :m1 run :m1 remember distance "blue fd :m2	**Example commands** **(turtle s along mirror line)** gray, meet "mirror remember toheading "mirror run :m1 run :m1

Figure A5.74:Task 5 MTG

A5.2.4 Final versions of computer constructions of Candy and Laurel (type I-I pairing, MTG-FO)

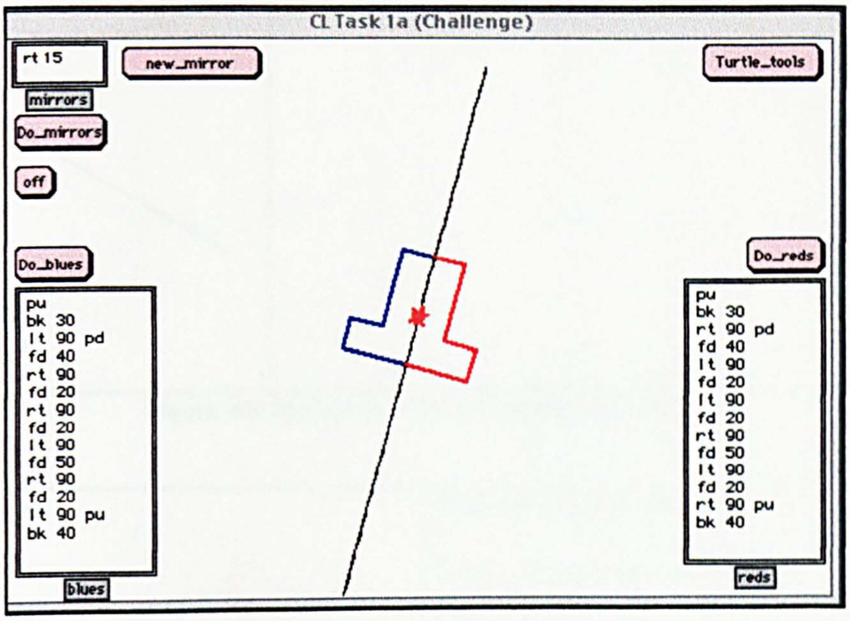


Figure A5.76:Task 1a MTG

Example figure

Procedures

```
to fred :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

```
to die :sh :hip
pu
fd 80 pd
rt 90 fd 10 rt 90
fd 20 rt 90 fd 10
lt 90
fd 10
lt :sh
fd 30 bk 30
rt :sh
fd 50
lt :hip
fd 50 bk 50
rt :hip
rt 180
end
```

Figure A5.77:Task 1b MTG

Example figure	Example commands

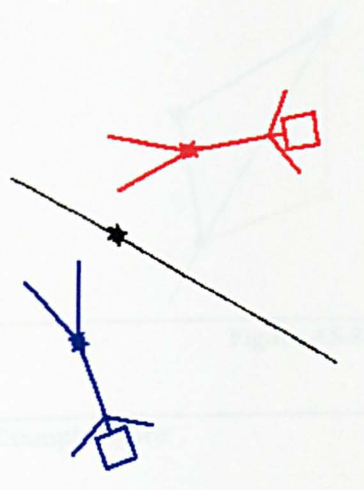
	`blue, lt  fd 70 lt 40`
	`pd fred 120 20 die 120 20`
	`red, rt  fd 70 rt 40`
	`pd die 120 20 fred 120 20`

Figure A5.78:Task 2 MTG (reflecting the person)

Example figure	Example commands

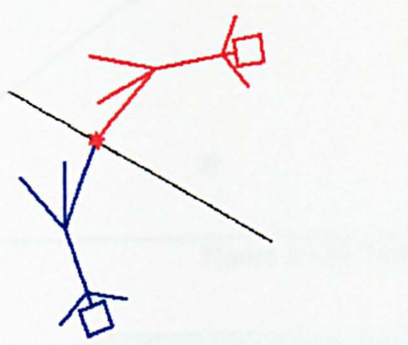
	`blue, show towards "mirror`
	`lt 140`
	`blue, show distance "mirror`
	`fd 70`
	`blue, show toheading "mirror`
	`lt 80`
	`red, rt 140 fd 70 rt 80`

Figure A5.79:Task 2 MTG (reuniting turtles, method 1)

	Example commands

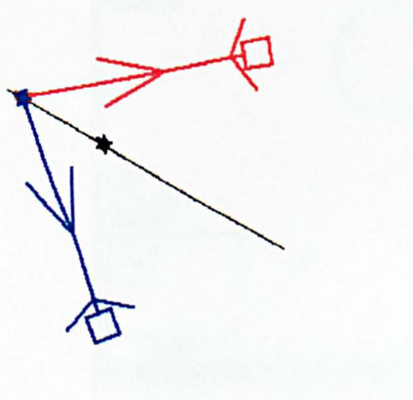
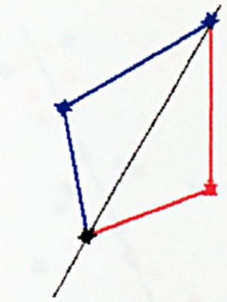
	`blue, meet "red`
	`blue, bk 107`
	`red, bk 107`

Figure A5.80:Task 2 MTG (reuniting turtles, method 2)

Example figure

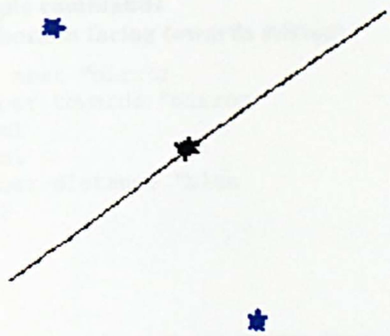


Example commands

```
blue, lt 40 fd 100 rt 70
meet "mirror
remember toheading "mirror
lt 30
remember distance "blue
blue, pd fd :m2 bk :m2
red, rt 40 fd 100 lt 70
pd fd :m2 bk :m2
```

Figure A5.81:Task 3 MTG (symmetrical quadrilateral)

Example figure



Procedure

```
to freda
blue, meet "mirror
remember towards "mirror
run :m1
run :m1
remember distance "blue
bk :m2
end
```

Figure A5.82:Task 3 MTG (image turtle)

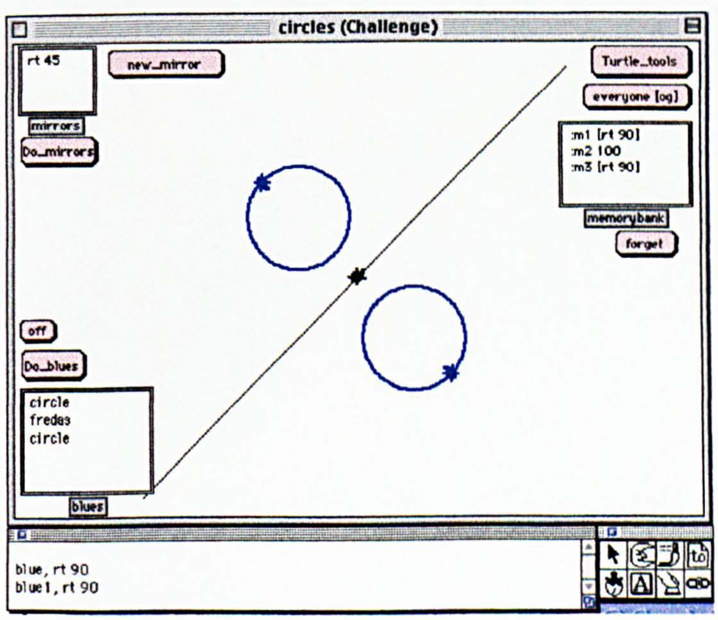
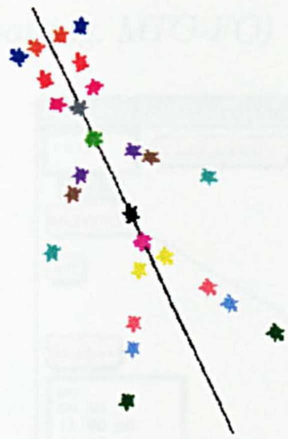


Figure A5.83:Task 4 MTG

Figure

	Example commands
	(turtle facing towards mirror)
	`pink, meet "mirror`
	`remember towards "mirror`
	`run :m1`
	`run :m1`
	`remember distance "pink`
	`bk :m2`

Example commands
(with bottom facing towards mirror)
`blue, meet "mirror`
`remember towards "mirror`
`run :m1`
`run :m1`
`remember distance "blue`
`fd :m2`
	Example commands
	(turtle s along mirror line)
	`gray, meet "mirror`
	`remember towards "mirror`
	`run :m1`
	`run :m1`

Figure A5.84:Task 5 MTG

A5.2.5 Final versions of computer constructions of Prija and Jodie (type II-III pairing, MTG-FO)

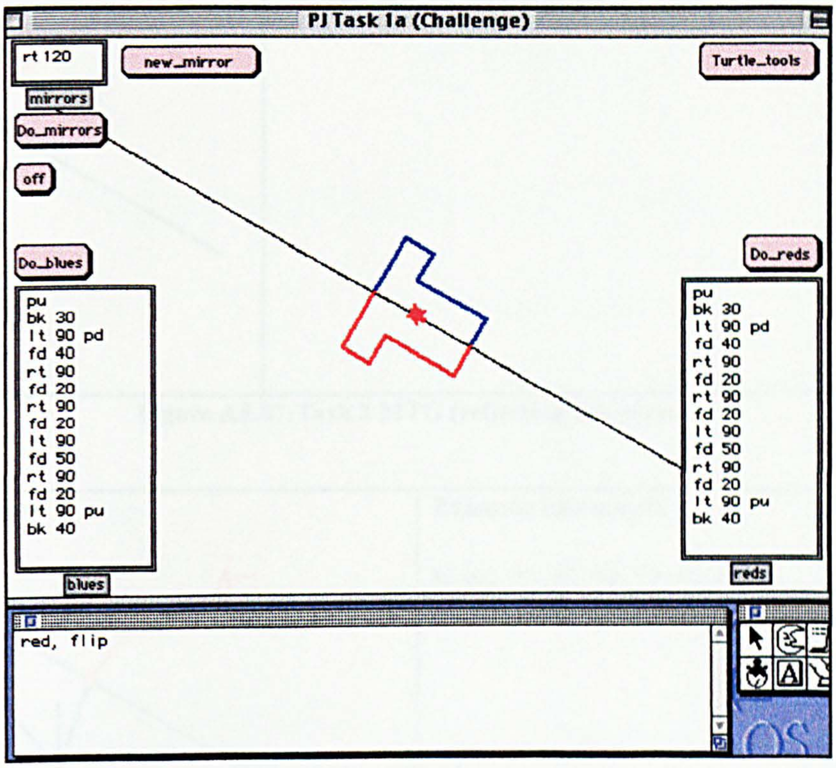


Figure A5.85:Task 1a MTG

Example figure

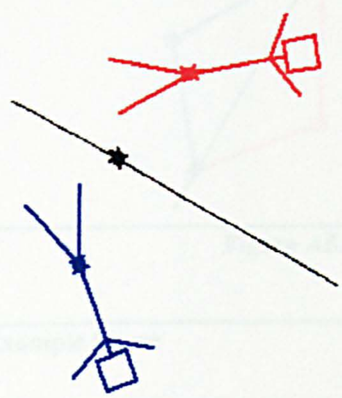
Procedures

```
to man :sh :hip
pu
fd 80 pd
lt 90 fd 10 lt 90
fd 20 lt 90 fd 10
rt 90
fd 10
rt :sh
fd 30 bk 30
lt :sh
fd 50
rt :hip
fd 50 bk 50
lt :hip
lt 180
end
```

```
to nam :hs :pih
pu
fd 80 pd
rt 90 fd 10 rt 90
fd 20 rt 90 fd 10
lt 90
fd 10
lt :hs
fd 30 bk 30
rt :hs
fd 50
lt : pih
fd 50 bk 50
rt : pih
rt 180
end
```

Figure A5.86:Task 1b MTG

Example figure

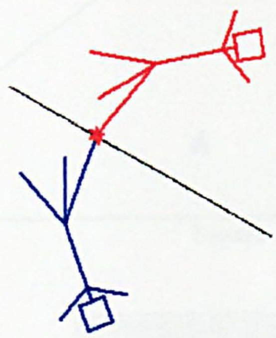


Example commands

```
blue, lt  fd 70 lt 40
pd man 120 20 nam 120 20
red, rt  fd 70 rt 40
pd nam 120 20 man 120 20
```

Figure A5.87:Task 2 MTG (reflecting the person)

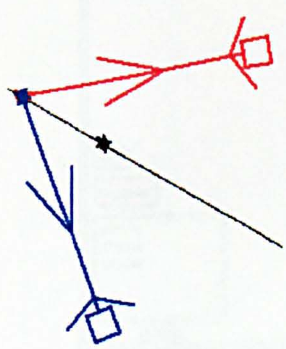
Example figure



Example commands

```
blue, rt 40 bk 70 rt 100
red, lt 40 bk 70 lt 100
```

Figure A5.88:Task 2 MTG (reuniting turtles, method 1)



Example commands

```
blue, meet "red
blue, bk 107
red, bk 107
```

Figure A5.89:Task 2 MTG (reuniting turtles, method 2)

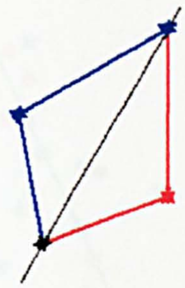
Example figure	Example commands
	blue, pd lt 40 fd 100 red, pd rt 40 fd 100 blue, rt 70 red, lt 70 blue, meet "red remember distance "red blue, fd :m1 bk :m1 red, :m1 bk :m1

Figure A5.90:Task 3 MTG (symmetrical quadrilateral)

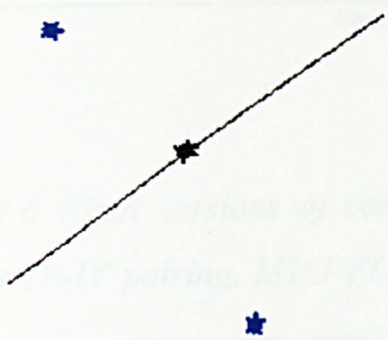
Example figure	Example commands
	blue, hatchhere remember towards "mirror rt 32 remember distance "mirror fd :m2 remember toheading "mirror lt 77 remember towards "blue rt 103 fd :m2 remember towards "mirror rt 180 rt 32

Figure A5.91:Task 3 MTG (image turtle)

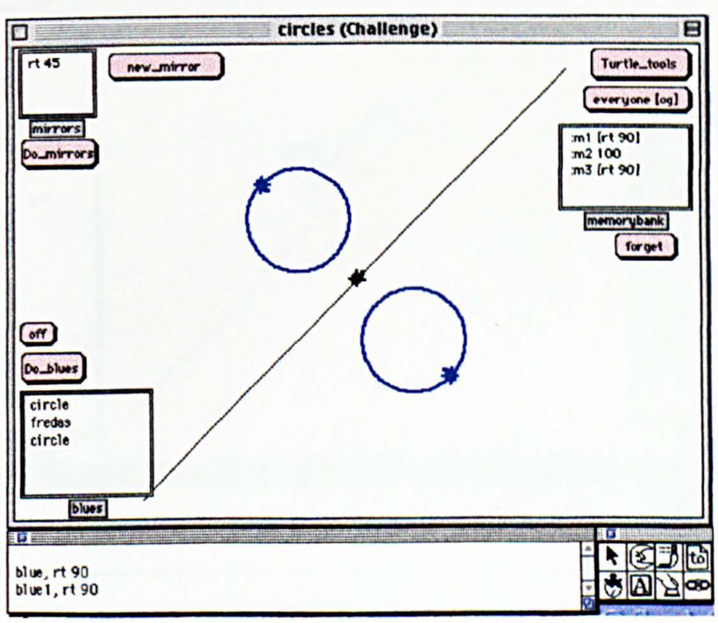


Figure A5.83:Task 4 MTG

Figure	Commands
	blue, hatchhere remember towards "mirror run :m1 remember distance "mirror fd :m2 remember toheading "mirror run :m3 remember towards "blue swop :m4 fd :m2 remember towards "mirror rt 180 **run :m1**

Figure A5.92:Task 5 MTG

A5.2.6 Final versions of computer constructions of Sophy and Kerry (type IV-IV pairing, MTG-FO)

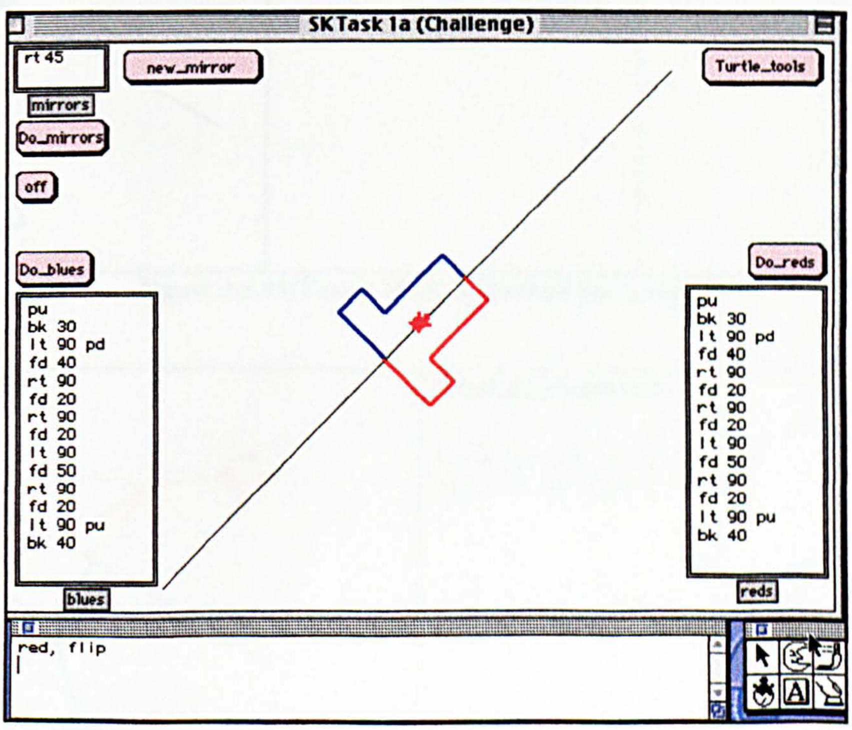


Figure A5.93:Task 1a MTG

Example figure	Procedures	
	to per :sh :hip pu fd 80 pd lt 90 fd 10 lt 90 fd 20 lt 90 fd 10 rt 90 fd 10 rt :sh fd 30 bk 30 lt :sh fd 50 rt :hip fd 50 bk 50 lt :hip lt 180 end	to son :sh :hip pu fd 80 pd rt 90 fd 10 rt 90 fd 20 rt 90 fd 10 lt 90 fd 10 lt :sh fd 30 bk 30 rt :sh fd 50 lt :hip fd 50 bk 50 rt :hip rt 180 end

Figure A5.94:Task 1b MTG

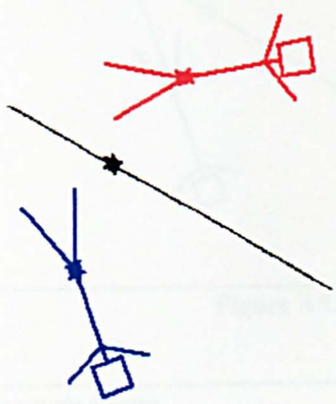
Example figure	Example commands	Procedures
	blue, gogo 100 70 40 pd per 120 20 son 120 20 red, ogog 100 70 40 pd per 120 20 son 120 20	to gogo :a :b :c lt :a fd :b lt :c end to ogog :a :b :c rt :a fd :b rt :c end

Figure A5.95:Task 2 MTG (reflecting the person)

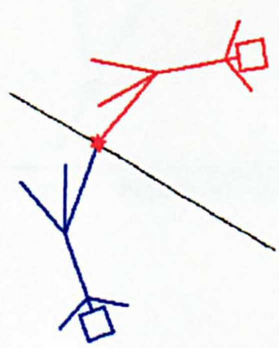
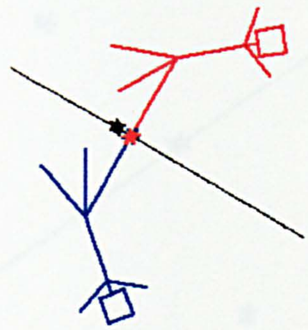
Example figure	Example commands
	blue, rt 40 bk 70 rt 100 red, lt 40 bk 70 lt 100

Figure A5.96:Task 2 MTG (reuniting turtles, method 1)

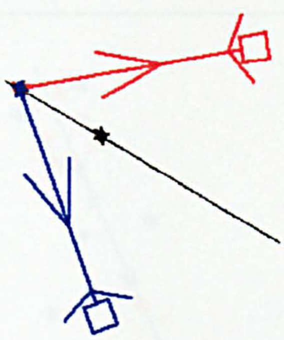
Example figure



Example commands

```
blue, show towards "red
lt 130
blue, show distance "red
fd 79
red, rt 130
fd 79
```

Figure A5.97:Task 2 MTG (reuniting turtles, method 2)

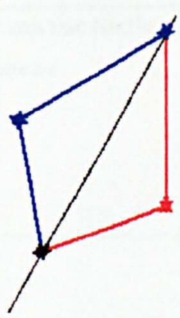


Example commands

```
blue, meet "red
blue, bk 107
red, bk 107
```

Figure A5.98:Task 2 MTG (reuniting turtles, method 3)

Example figure



Example commands

```
blue, pd lt 40 fd 100
red, pd rt 40 fd 100
blue, rt 70
red, lt 70
blue, meet "red
remember distance "blue
remember toheading "mirror
lt 30
blue, fd 107 bk 107
red, fd 107 bk 107
```

Figure A5.99:Task 3 MTG (symmetrical quadrilateral)

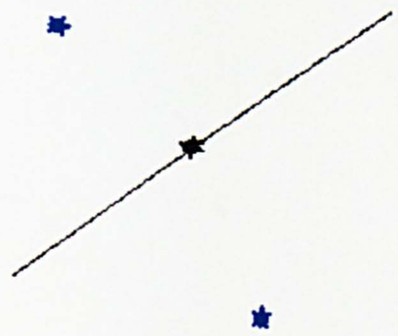
<table>
<tr><th>Example figure</th><th>Example commands</th></tr>
<tr><td></td><td>mirror, hatchhere
show towards "blue
rt 103
show distance "blue
fd 103
mirror, hatchhere
lt 103 fd 103
show toheading "blue
<delete second new turtle then click on the image turtle>
lt 148</td></tr>
</table>

Figure A5.100:Task 3 MTG (image turtle)

<table>
<tr><th>Figure</th><th>Example commands
(turtle not on top of mirror line)</th></tr>
<tr><td></td><td>mirror, hatchhere
show towards "blue
rt 10
show distance "blue
fd 147
mirror, hatchhere
lt 10 fd 147
show towards "blue
<delete second new turtle then click on the image turtle>
rt 20</td></tr>
<tr><td>Commands for mirror turtle
mirror, hatchhere</td><td>Example commands
(turtle s along mirror line)
mirror, hatchhere
show distance "gray
fd 90
show towards "gray
rt 56</td></tr>
</table>

Figure A5.101:Task 5 MTG